Sets and Functions [1]

1 Sets

A **set** is a collection of **elements**. The expression $p \in S$ means p is an element of the set S. A set may be defined in several ways: in ordinary English, *e.g.,* let A be the set of positive even integers; by listing its elements within braces, *e.g.,* let $A = \{2, 4, 6, 8, ...\}$; or by using "set builder" notation, *e.g.,* $A = \{n \in \mathbb{Z} \,|\, n > 0 \text{ and } n \text{ is even}\}$, read: A is the set of all integers n such that $n > 0$ and n is even ($\mathbb{Z}$ is the standard notation for the integers).

A set does not have an order. Thus $\{a, b\} = \{b, a\}$. An **ordered set** is a set together with an ordering. When we want to stress that a set has been endowed with an ordering we will use parenthesizes instead of braces: (a, b) is an ordered set and is not equal to (b, a).

The following notations are standard:

- $\phi = \{\}$, the empty set.
- $A \subset B$: read A is a subset of B, meaning, every element of A is an element of B. *Example:* $\{2, 5\} \subset \{1, 2, 3, 4, 5\}$.
- $A \cup B$: A union B, meaning, the set of all elements that are in A **or** in B. *Example:* $\{\$, *, !\} \cup \{\alpha, !, \star, 17\} = \{\$, *, !, \alpha, \star, 17\}$.
- $A \cap B$: read A intersection B, meaning, the set of all elements that are in A **and** in B. *Example:* $\{\$, *, !\} \cap \{\alpha, !, \star, 17\} = \{!\}$.
- $A - B$: read A minus B, meaning, the set of all elements of A that are not elements of B. *Example:* $\{\$, *, !\} - \{\alpha, !, \star, 17\} = \{\$, *\}$.
- $A \times B$: read A cross (product) B, meaning, the set of ordered pairs (a, b) where $a \in A$ and $b \in B$. Since there is a natural one-to-one correspondence between $(A \times B) \times C$ and $A \times (B \times C)$, $((a, b), c) \longleftrightarrow (a, (b, c))$, we shall ignore the distinction between them and use the notation $A \times B \times C$ for the set $\{(a, b, c) \,|\, a \in A, b \in B, \text{ and } c \in C\}$. Other multiple cross products are defined similarly. *Examples:* $\{1, 3\} \times \{0, 1, 2\} = \{(1, 0), (1, 1), (1, 2), (3, 0), (3, 1), (3, 2)\}$. $\{*, \#\} \times \{\%\} = \{(*, \%), (\#, \%)\}$.

- $A^n = A\times\cdots\times A$, n times. *Example:* $\{2,3\}^3 = \{(2,2,2),(2,2,3),(2,3,2),(2,3,3),(3,2,2),(3,2,3),(3,3,2),(3,3,3)\}$.

Some standard sets are:

- $\mathbb{Z}$: the integers (from the German *zummer*)
- $\mathbb{Q}$: the rationals (quotients)
- $\mathbb{R}$: the reals
- $\mathbb{C}$: the complex numbers

Remark. The sets $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are normally given an ordering. Interestingly, $\mathbb{C}$ is not typically ordered.

Interval Notation:

$$\begin{array}{rcl rcl}
[a,b] &=& \{x\in\mathbb{R}\,|\,a\le x\le b\} & [a,\infty) &=& \{x\in\mathbb{R}\,|\,a\le x\}\\
(a,b) &=& \{x\in\mathbb{R}\,|\,a< x< b\} & (a,\infty) &=& \{x\in\mathbb{R}\,|\,a< x\}\\
(a,b] &=& \{x\in\mathbb{R}\,|\,a< x\le b\} & (-\infty,b] &=& \{x\in\mathbb{R}\,|\,x\le b\}\\
[a,b) &=& \{x\in\mathbb{R}\,|\,a\le x< b\} & (-\infty,b) &=& \{x\in\mathbb{R}\,|\,x< b\}
\end{array}$$

Remark. The notation "(a,b)" is ambiguous; it could represent an interval or an ordered pair. One has to consider the context to understand the intended meaning. On behalf of mathematicians everywhere, I apologize for any in convenience this may cause.

Examples:

- $\{x\in\mathbb{R}\,|\,x\le-\sqrt{7}\}\cup\{x\in\mathbb{R}\,|\,x\ge\sqrt{7}\}$ is the solution set for $x^2-7\ge 0$.
- $\mathbb{R}-\{0\}$ is the natural domain of $1/x$.
- $\mathbb{R}^2$ is the plane. $\mathbb{R}^3$ is 3-dimensional space. $\mathbb{R}^4$ is 4-dimensional space. And so on.
- $\phi\subset A$, $\phi = A\cap\phi$, and $A = A\cup\phi$ are true statements for all sets A.
- $\{x\in\mathbb{R}\,|\,-2\le x<5\} = [-2,5) = [-2,7]\cap(-10,5)$.
- $S=[0,1]\times[0,1]$ is the *unit square* in the plane $\mathbb{R}^2$ with corners (0,0), (1,0), (0,1), and (1,1).

Problems:

1. Describe $[0,1] \times [0,2] \times [0,3]$.

2. Simplify $((1,3) \cap (2,5)) \cup [3,4)$.

3. Let $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9\}$, $B = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$, and $C = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$. Graph $A - B$, $A \cap (\mathbb{R}^2 - B)$, $A \cap C$, and $A - C$.

4. Find the solution set in $\mathbb{R}^2$ of $\sin x \cos y = 0$.

5. Draw $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times \mathbb{R}$, and $((0,1] \cup \{2,3\}) \times ([-2,-1] \cup (2,3))$ as subsets of $\mathbb{R}^2$.

6. Let A be a set. What is $A \times \phi$?

7. [Hard] Let A, B, and C be sets. Prove that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. (You can draw pictures to "see" this, but you need to reason from the definitions to prove it.)

2 Functions

Intuitively, a function f from a set A to a set B assigns to each element of A one element of B. Formally, f is a subset of $A \times B$ such that for every $a \in A$ there is one and only one $b \in B$ with $(a,b) \in f$. We normally write $f : A \to B$, and express $(a,b) \in f$ by $b = f(a)$.

A function $f : A \to B$ is **onto** if for every $b \in B$ there is at least one $a \in A$ such that $(a,b) \in f$, *i.e.*, such that $f(a) = b$. A function $f : A \to B$ is **one-to-one** if for every $b \in B$ there is at most one $a \in A$ with $f(a) = b$.

Let $f : A \to B$, $A' \subset A$, and $B' \subset B$. Then we define,

- $f(A') = \{b \in B \mid b = f(a) \text{ for at least one } a \in A'\}$ and is called the **image** of A' under f. We call $f(A)$ the **range** of f.

- $f^{-1}(b) = \{a \in A \mid b = f(a)\}$.

- $f^{-1}(B') = \{a \in A \mid a \in f^{-1}(b) \text{ for at least one } b \in B'\}$

If f is one-to-one and onto then $f^{-1}(b)$ always consists of a single element and we regard f^{-1} as a function from B to A. In this case we say f is **invertible**.

A **binary operation** is a function from the cross product of two sets to a third set. For example, the adding of two numbers is a binary operation from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. So is multiplication. For any binary operation $f : A \times B \to C$, if $a_1 = a_2 \in A$ and $b \in B$ then $f(a_1, b) = f(a_2, b)$. For multiplication this means for real numbers a, b, and c, if $a = b$ then $ac = bc$. Note that we have written $f(a, b)$ instead of $f((a, b))$ since this shorthand is customary.

Example 1. Let $S = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit, \square, \bigcirc, \star\}$ and let $L = \{\alpha, \theta, \phi, \pi, \zeta\}$. Let $f : S \to L$ be defined as indicated by Figure 1. But what *is* f really? It is the set of arrows. But each arrow is a pictorial representative of an ordered pair. Thus $(\clubsuit, \alpha) \in f$ but $(\diamondsuit, \zeta) \notin f$. Or, equivalently, $f(\clubsuit) = \alpha$ while $f(\diamondsuit) \neq \zeta$. This function is not one-to-one since, for example, $f(\clubsuit) = f(\bigcirc)$. It is not onto since there is no $x \in S$ such that $f(x) = \zeta$, that is, for every $x \in S$, $(x, \zeta) \notin f$. Or, we could say ζ is not in the range of f.

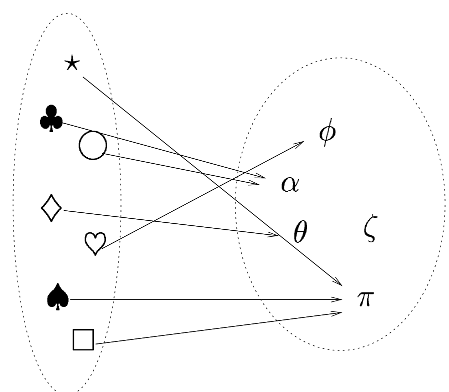

Figure 1: A function

If we order the elements of S and L then we can **graph** f. This is shown in Figure 2. We can see that the graph of f is a subset of $S \times L$. Notice that the familiar *horizontal line test* shows that f is not one-to-one, while the *vertical line test* confirms that f is indeed a function.

Additional Examples:

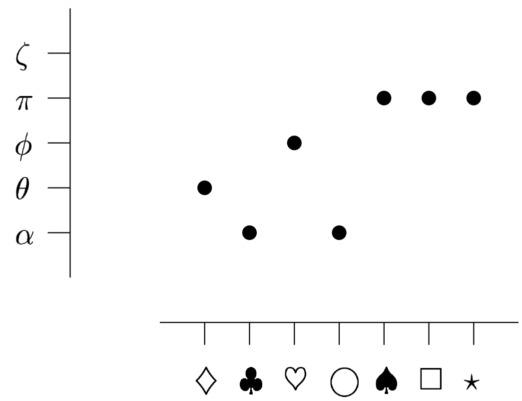

Figure 2: A graph of the function in Figure 1

- Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Then $f^{-1}(4) = \{-2, 2\}$, $f^{-1}([0,1]) = [-1, 1]$, and $f^{-1}([1,9]) = [-3,-1] \cup [1,3]$.

- Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin \pi x$. Then $f^{-1}(0) = \mathbb{Z}$, and

$$f^{-1}([0,1]) = \cdots \cup [-4,-3] \cup [-2,-1] \cup [0,1] \cup [2,3] \cup \cdots$$

- Let $A = \{1, 2, 3, ...\}$. Then the set $\{(1,2), (2,3), (3,4), ...\} \subset A \times A$, is the function $f : A \to A$ produced by adding a one: $f(n) = n + 1$. It is one-to-one but not onto. But if we let $B = A - \{1\}$ and let $g : A \to B$ be addition by one, then g is onto.

- The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is the set $\{(x,y) \in \mathbb{R}^2 \mid y = x^2\}$. Thus, you can think of the function f as the graph in the plane $\mathbb{R}^2$.

- Let $A = \{2, 3\}$. Let $f = \{(2,3), (3,3)\}$, $g = \{(2,3), (3,2)\}$, and $h = \{(2,2), (2,3)\}$. Then, f is a function from A to A that is not one-to-one or onto, g is a one-to-one onto function from A to A, while h is not be a function. Check that $g^{-1}(f(3)) = 2$ and that $f(g(f(x))) = g(f(g(x)))$ for all $x \in A$.

Problems:

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x^2 + y^2$. Draw a picture of $f^{-1}([4,9])$. *Recall:* $[4,9] \subset \mathbb{R}$ is the closed interval from 4 to 9. *Hint:* What is $f^{-1}(4)$?

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \sin x \cos y$. Find $f^{-1}(0)$ and $f^{-1}(1)$. Draw pictures of them.

3. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f(x) = (x, x^2)$. Show that f is one-to-one but not onto.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x + y, x + y)$. Show that f is neither one-to-one nor onto. Describe the range of f.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (3x + 2y, x - y)$. Show that f is one-to-one and onto. Find f^{-1}. What is the image of $\{(x, y) \in \mathbb{R}^2 \mid x = y\}$?