[OPTIONAL READING]

An alternative approach to the derivatives of the natural log and exponential functions

Calculus 150

Let $e = \lim_{c \to 0}(1+c)^{1/c}$. We assume this limit exists. Consider the following limit where x is a nonzero real number: $\lim_{c \to 0}(1+c/x)^{1/c}$. Let $a = c/x$. Then $c = ax$. As c goes to zero so does a. Hence

$$\begin{aligned}
\lim_{c \to 0}(1+c/x)^{1/c} &= \lim_{a \to 0}(1+a)^{1/ax} \\
&= \lim_{a \to 0}[(1+a)^{1/a}]^{1/x} \\
&= [\lim_{a \to 0}(1+a)^{1/a}]^{1/x} \\
&= e^{1/x}.
\end{aligned}$$

Define $\ln(x)$ to be the inverse of e^x. We assume both these functions are continuous. Then

$$\begin{aligned}
(\ln x)' &= \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h} \\
&= \lim_{h \to 0} (1/h) \ln\left(\frac{x+h}{x}\right) \\
&= \lim_{h \to 0} \ln(1+h/x)^{1/h} \\
&= \ln\left(\lim_{h \to 0}(1+h/x)^{1/h}\right) \\
&= \ln\left(e^{1/x}\right) \\
&= 1/x.
\end{aligned}$$

By the Chain Rule one can show that for a differentiable function $f(x)$ the derivative of f inverse is given by $(f^{-1}(x))' = 1/f'(f^{-1}(x))$. We can apply this to e^x which is the inverse of $\ln x$. This gives $(e^x)' = 1/(1/e^x) = e^x$.

Problem: Show that $(\ln|x|)' = 1/x$ for both positive and negative x. This variation will come up in many integration problems.