The Power Rule and Polynomials [1]

Theorem .1 (Power Rule for Positive Integers).

$$(x^n)' = nx^{n-1}.$$

We already know the theorem is true for low values of n, specifically $n = 1, 2, 3, 4, 5$, from earlier work. Suppose we knew it for say $n = 103$. That is, suppose we had checked that

$$(x^{103})' = 103x^{102}.$$

Can we show the formula holds for $n = 104$? Yes.

$$\begin{aligned}
(x^{104})' &= (x \cdot x^{103})' \\
&= (x)'(x^{103}) + x(x^{103})' \\
&= x^{103} + x(103x^{102}) \\
&= x^{103} + 103x^{103} \\
&= 104x^{103}.
\end{aligned}$$

It seems then that we could go forever. The proposition that we can indeed make this jump is called the **principle of mathematical induction**. The idea is much less foreboding than the name. The proof given below uses induction in a formal manner.

Proof. We know that the formula is true for $n = 1$. Suppose for the moment that we know the formula is true for some particular value of $n \geq 1$; call it $\hat{n}$. That is we are given

$$(x^{\hat{n}})' = \hat{n}x^{\hat{n}-1},$$

not for all $\hat{n}$, but just for a particular $\hat{n}$. We will show that under this supposition the formula is true for $n = \hat{n} + 1$.

$$\begin{aligned}
(x^{\hat{n}+1})' &= (x \cdot x^{\hat{n}})' \\
&= (x)'(x^{\hat{n}}) + x(x^{\hat{n}})' \\
&= x^{\hat{n}} + x(\hat{n}x^{\hat{n}-1}) \\
&= x^{\hat{n}} + \hat{n}x^{\hat{n}} \\
&= (\hat{n}+1)x^{\hat{n}}.
\end{aligned}$$

The principle of mathematical induction allows us to conclude that the formula must hold any positive integer n. The proof of our theorem is complete. □

Problem 1 (Power Rule for Negative Integers). The Power Rule works for negative integers. We have shown this is the case for x^{-1}. Use induction to prove the Power Rule works for any negative integer.

Remark (The case $n = 0$). If $n = 0$ and we applythe Power Rule we would get $0x^{-1}$. This is not defined at zero, but then neither is x^0. For nonzero values of x we get then $x^0 = 1$, a constant, whose derivative is the zero function. Thus, we can write $(x^0)' = 0x^{-1}$ So, the Power Rule works for all integers, even zero.

Problem 2. Find a general formula for $(f^n(x))'$. Use induction to prove its correctness.

Problem 3 (Philosophical). Suppose x is distance in meters. Then x^3 is meters cubed, a measure of volume. Further, $3x^2$ has units of meters squared. But shouldn't the derivative of x^3 be in units of volume per time, the rate of change of a volume? Resolve this seeming paradox.

Project 1. Let n be a positive integer. Use mathematical induction to prove that $(\sin nx)' = n \cos nx$ and $(\cos nx)' = n \sin nx$. Hint: They must be done simultaneously.

Polynomials: We can apply the Power Rule and the Sum Rule to find the derivative of any polynomial.

Theorem .2. *Let* $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. *Then*

$$p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1.$$

Examples.

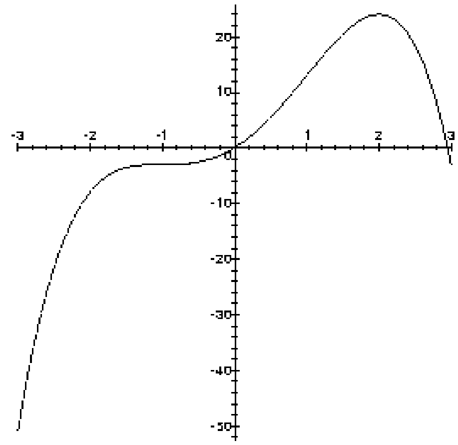
1. $(2x^3 + 5x^2 - x + 56)' = 6x^2 + 10x - 1$.
2. $(x^6 + 3x^4 - 5x)' = 6x^5 + 12x^3 - 5$.
3. $(x^{-3} + 3x^{-1})' = -3x^{-4} - 3x^{-2}$. Note: not polynomials.
4. $(x^{-1} + 3x^4 + 7x)'' = 2x^{-3} + 36x^2$. Check this!

Application: Finding the maximum of a function.

Let $f(x) = -x^4 + 6x^2 + 8x$. Find the maximum value of $f(x)$. Solution: The -1 coefficient of x^4 forces $f(x)$ towards $-\infty$ for large magnitudes of x. Thus, $f(x)$ should have a maximum value. At a point $x = c$ where $f(x)$ is largest we should have $f'(c) = 0$. Why? See the graph below. Now $f'(x) = -4x^3 + 12x + 8 = -4(x+1)^2(x-2)$. Thus, $f'(x) = 0$ for $x \in \{-1, 2\}$. We can check that $f(2) = 24$ is bigger then $f(-1) = -3$. Thus, $f(x)$ has a maximum of 24 at $x = 2$. Question: What is happening at $x = -1$, the other zero of $f'(x)$?

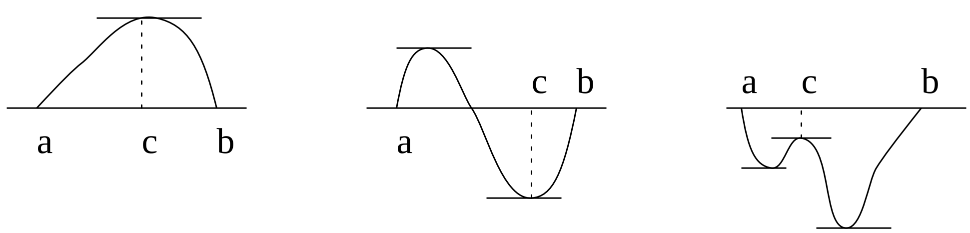
Problems.

1. Find the second derivative of $f(x) = 3x^{-2} + x^3 \sin x$
2. Find a function $f(x)$ such that $f'(x) = 3x^2 + 4x$.
3. Find a function $f(x)$ such that $f'(x) = 3 + x^{-2} + 6x^2$.
4. Find several functions which have second derivative x^3.
5. Find the maximum value of $f(x) = (1 - x^2)\sqrt{x}$ for $x > 0$.
6. Find the minimum value of $f(x) = x(1+x)^2$ for $x \in [-3, 3]$.



Project 2. Let $f(x)$ be a differentiable function. Suppose $f(a) = f(b) = 0$ with $a < b$. Then it should be clear that for at least one number, call it c, between a and b, we must have $f'(c) = 0$. See the figure below. This fact is known as **Rolles' Theorem**.

Let n be a positive integer. Use Rolles' Theorem and mathematical induction to show that if $p(x)$ is a polynomial of degree n then $p(x)$ has at most n real zeros.



Problem: The following problem is of great importance. But don't be too frustrated if you can't get it now. We will come back to it later. Find a function whose derivative is $1/x$.