The Product Rule [1]

Suppose $f(x)$ is the product of two functions, $g(x)$ and $h(x)$, whose derivatives we already know. What is the derivative of $f(x)$? That is, if

$$f(x) = g(x)h(x),$$

where $g'(x)$ and $h'(x)$ are known, what is $f'(x)$? A natural first guess might be

$$f'(x) = g'(x)h'(x).$$

Let's test it. Suppose $f(x) = x^2$. We can let $g(x) = h(x) = x$. But then we would get

$$(x^2)' = (x)'(x)' = 1 \times 1 = 1.$$

But this is false, it contradicts what we already have shown. Here is an even more damaging example. Let $f(x) = x$ and then let $g(x) = x$ and $h(x) = 1$. Then we would get

$$1 = (x)' = (x \times 1)' = (x)'(1)' = 1 \times 0 = 0.$$

Here is another difficulty with our initial conjecture. Suppose g and h are functions of time in seconds and have units of length, say feet. Then g' and h' have units of ft/sec. Further, f has units of area, ft^2. Then f' should have units of area per time, ft^2/sec. That is f' is a rate of change of area. But the product $g'(x)h'(x)$ has units of ft^2/sec^2. So, the units don't match up.

What then is the right formula? We will turn to geometry for help. Suppose as above that $f(x) = g(x)h(x)$ and that x is time in seconds and g and h are in feet. Then we can think of $f(x)$ as the area of a rectangle with sides of length $g(x)$ and $h(x)$. See the figure below. Then $f'(x)$ is the rate of change of the area of this rectangle. Suppose that x, time, changes by an amount Δx. Suppose further that we know how much g and h change during the time lapse of Δx. We write these as

$$\Delta g = g(x + \Delta x) - g(x), \text{ and } \Delta h = h(x + \Delta x) - h(x).$$

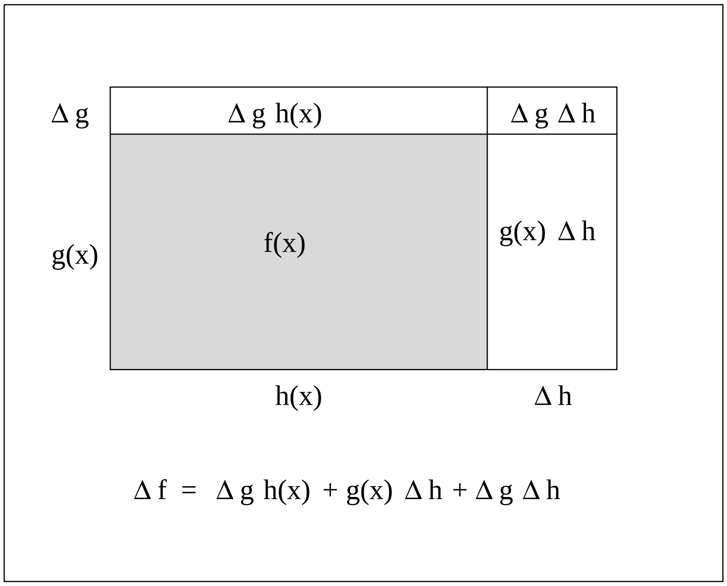

Figure 1: A Dynamic Rectangle

The shaded smaller rectangle represents f at time x while the larger rectangle represents f at time $x + \Delta x$. From the figure it seems clear that $f(x + \Delta x)$ is just the sum of its parts. This gives

$$f(x + \Delta x) = f(x) + \Delta g\, h(x) + \Delta h\, g(x) + \Delta g \Delta h.$$

Now we can compute Δf.

$$\Delta f = f(x + \Delta x) - f(x) = \Delta g\, h(x) + \Delta h\, g(x) + \Delta g \Delta h. \tag{1}$$

Since by definition $f'(x) = \lim_{\Delta x \to 0} \Delta f / \Delta x$ we divide both sides by Δx and take the limit as $\Delta x \to 0$. This gives

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{\Delta g\, h(x)}{\Delta x} + \frac{\Delta h\, g(x)}{\Delta x} + \frac{\Delta g \Delta h}{\Delta x} \right].$$

Rearranging the fractions on the right hand side a bit gives

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \left[h(x) \frac{\Delta g}{\Delta x} + g(x) \frac{\Delta h}{\Delta x} + \Delta g \frac{\Delta h}{\Delta x} \right].$$

Using standard properties of limits we can get

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = h(x) \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} + g(x) \lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x} + \lim_{\Delta x \to 0} \Delta g \lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x}.$$

Now using the definition of derivatives and the obvious, but subtle, fact that

$$\lim_{\Delta x \to 0} \Delta g = 0,$$

we get

$$f'(x) = h(x)g'(x) + g(x)h'(x) + 0 \cdot h'(x),$$

which can of course be written as in the box below.

$$\boxed{f'(x) = g'(x)h(x) + g(x)h'(x)}$$

This equation is called the **Product Rule**. It is the formula we seek.

But, pictures can be deceiving. Our figure for example implicitly assumes that g and h are positive and increase with x. However, the argument above can be turned into a completely formal proof by noting that formula (1) for Δf can be arrived at as follows.

$$\begin{aligned} \Delta f &= f(x + \Delta x) - f(x) \\ &= g(x + \Delta x)h(x + \Delta x) - g(x)h(x) \\ &= (g(x) + \Delta g)(h(x) + \Delta h) - g(x)h(x) \\ &= \Delta g\, h(x) + g(x)\Delta h + \Delta g \Delta h. \end{aligned}$$

The product rule follows.

Example 1. Find the derivative of $x \sin x$.

Solution.

$$(x \sin(x))' = (x)' \sin x + x(\sin x)' = \sin x + x \cos x$$

□

Example 2. Find the derivative of x^2 using the Product Rule.

Solution.

$$(x^2)' = (x)'x + x(x)' = x + x = 2x$$

$\square$

Example 3. Find the derivative of $x^{3/2}$. Assume $x > 0$.

Solution. First we rewrite $x^{3/2}$ as $x\sqrt{x}$.

$$(x\sqrt{x})' = (x)'\sqrt{x} + x(\sqrt{x})' = \sqrt{x} + x\left(\frac{1}{2\sqrt{x}}\right) = \frac{3}{2}x^{\frac{1}{2}}$$

$\square$

Example 4. Find the derivative of $\sin 2x$ using the Product Rule rather than the Baby Chain Rule.

Solution.

$$\begin{aligned}
(\sin 2x)' &= (2\sin x\cos x)' \\
&= 2((\sin x)'\cos x + \sin x(\cos x)') \\
&= 2(\cos^2 x - \sin^2 x) \\
&= 2\cos 2x
\end{aligned}$$

$\square$

Example 5. Find the derivative of $(x^3+1)(7x+3)$.

Solution.

$$\begin{aligned}
((x^3+1)(7x+3))' &= (x^3+1)'(7x+3) + (x^3+1)(7x+3)' \\
&= 3x^2(7x+3) + 7(x^3+1) \\
&= 28x^3 + 9x^2 + 7
\end{aligned}$$

$\square$

Problems. Find the derivatives of the functions in problems 1-16 using the Product Rule.

(1) x^3

(2) x^4

(3) x^5

(4) $x^{\frac{5}{2}}$

(5) $\sin^2 x$

(6) $\sin^3 x$

(7) $\cos^4 x$

(8) $x \sin x \cos x$

(9) $x^2 \sin x$

(10) $\cos 2x$

(11) $\sin 3x$

(12) $x^2 \cos^2$

(13) $(x^3 + x - 5)(x^2 - 1)$

(14) $(x^2 + 1) \sin x$

(15) $\dfrac{x^2 + \cos^2 x}{3x}$

(16) $\dfrac{\sin x}{x}$

(17) Let c be a constant and $f(x)$ a differentiable function. Use the product rule to show that $(cf(x))' = cf'(x)$.

(18) If $f(x)$, $g(x)$, and $h(x)$ are differentiable functions find a formula for $(fgh)'$. Explain your formula intuitively/geomertically using a rectangular box with side lengths f, g and h.

(19) [Computer required] Graph $x \sin x$ for $x \in [-2\pi, 2\pi]$. Graph $(x \sin x)'$ below it using the same scale. For x equals -4, -1 and 2 draw the tangent lines onto the first graph. Compute their slope by hand, showing your work. Compare them with the y values of the cooresponding x values of the second graph.