Math 150 Test 1 Fall 2010

Name: Key

CALCULATORS ALLOWED

1. [5 points] What is the domain of $\frac{x}{\sqrt{x-3}}$? Express your answer in interval notation.

We need $x-3>0$. Thus $x>3$, or

$$(3,\infty)$$

2. [5 points] Find the exact value of $\csc^{-1} 2$. Express your answer in radians.

Let $\theta = \csc^{-1} 2$.
Then $\csc\theta = 2$,
Thus $\sin\theta = \frac{1}{2}$ and so $\boxed{\theta = \frac{\pi}{6}.}$

3. [10 points] Let $f(x) = x^2 + \sqrt{x}$. Find the equation in slope-intercept form of the line tangent to the graph of $y = f(x)$ at the point (4,18).

$f'(x) = 2x + \frac{1}{2\sqrt{x}}$. $f'(4) = 8 + \frac{1}{4} = \frac{33}{4}$.

This is the slope. The line with slope m that goes through (a,b) is given by

$$y - b = m(x-a).$$

So we have $y - 18 = \frac{33}{4}(x-4)$

$$y = \frac{33}{4}x - 33 + 18$$

$$\boxed{y = \frac{33}{4}x - 15}$$

4. [20 points] Find the following limits. Show the steps you are using. Do not just plug in numbers.

$\frac{0}{0}$

a. $\lim_{x\to 0} \frac{\sin 2x}{\sin 4x} = \lim_{x\to 0} \frac{\sin 2x}{1} \cdot \frac{1}{\sin 4x}$

$= \lim_{x\to 0} \frac{2x}{4x} \cdot \frac{\sin 2x}{2x} \cdot \frac{4x}{\sin 4x} = \frac{1}{2} \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right) \cdot \lim_{x\to 0}\left(\frac{4x}{\sin 4x}\right)$

$= \frac{1}{2} \cdot 1 \cdot \frac{1}{1} = \boxed{\frac{1}{2}}$

b. $\lim_{x\to 3} \frac{x^2+x-12}{x^2-2x-3}$

If you plug in $x=3$ you get $\frac{0}{0}$ So we factor.

$= \lim_{x\to 3} \frac{(x+4)(x-3)}{(x+1)(x-3)}$

$= \lim_{x\to 3} \frac{x+4}{x+1} = \frac{3+4}{3+1} = \boxed{\frac{7}{4}}$

"$\frac{0}{0}$"

c. $\lim_{x\to 0} \frac{x}{x+\tan x} = \lim_{x\to 0} \frac{1}{\frac{x+\tan x}{x}}$

$= \lim_{x\to 0} \frac{1}{1+\frac{\tan x}{x}} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$

Since $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to} \frac{\sin x}{x} \frac{1}{\cos x} = 1\cdot 1.$

Another way: $\lim_{x\to 0} \frac{x}{x+\tan x} \cdot \frac{\frac{\cos x}{x}}{\frac{\cos x}{x}}$

$= \lim_{x\to 0} \frac{\cos x}{\cos x + \frac{\sin x}{x}} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$

"$\infty - \infty$"

d. $\lim_{x\to\infty} \frac{\sqrt{9x^2+x}-3x}{1} \cdot \frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x} =$

$\lim_{x\to\infty} \frac{9x^2+x-9x^2}{\sqrt{}+3x} = \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x}+3x} \frac{\frac{1}{x}}{\frac{1}{x}}$

$= \lim_{x\to\infty} \frac{1}{\sqrt{9+\frac{1}{x}}+3} = \frac{1}{\sqrt{9+0}+3} = \boxed{\frac{1}{6}}$

5. [5 points] Let $f(x)$ and $g(x)$ be differentiable functions. Prove that $(f(x)+g(x))' = f'(x)+g'(x)$.

$(f(x)+g(x))' = \lim_{h\to 0} \frac{f(x+h)+g(x+h) - (f(x)+g(x))}{h} = \lim_{h\to 0} \frac{f(x+h)-f(x) + g(x+h) - g(x)}{h}$

$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to \infty} \frac{g(x+h)-g(x)}{h} = f'(x)+g'(x)$, ~~[illegible]~~

both limits exist since $f(x)$ and $g(x)$ are differentiable

6. [30 points] Find the following derivatives.

a. $(x^2 \sin x)' =$

$(x^2)' \sin x + x^2 (\sin x)' =$

$2x \sin x + x^2 \cos x$

b. $(x^4 - 2x^2 + 3\tan x)' =$

$4x^3 - 4x + 3\sec^2 x$

c. $\left(\dfrac{7x+1}{x+3}\right)' =$

$$\frac{(7x+1)'(x+3) - (7x+1)(x+3)'}{(x+3)^2}$$

$$= \frac{7(x+3) - (7x+1)\cdot 1}{(x+3)^2}$$

$$= \frac{7x+21-7x-1}{(\quad)^2} = \frac{20}{(x+3)^2}$$

d. $(\sqrt{x} \cdot \cos x)' =$

$\frac{1}{2\sqrt{x}} \cos x - \sqrt{x} \sin x$

e. $(\cot 3x)' =$

$-\csc^2(3x)\cdot(3x)'$

$= -3\csc^2(3x)$

f. $(\sin^3 x)' = 3\sin^2 x \cdot (\sin x)'$

$= 3\sin^2 x \cos x$

7. [15 points] Let $f(x) = \arctan\left(\frac{1}{x}\right)$. Find the following limits – use any method.

a. $\lim_{x\to 0^+} f(x)$

As $x \to 0^+$, $\frac{1}{x} \to \infty$. Let $y = \frac{1}{x}$. $\lim_{y\to\infty} \arctan y = \boxed{\frac{\pi}{2}}$

b. $\lim_{x\to 0^-} f(x)$

As $x \to 0^-$, $\frac{1}{x} \to -\infty$. Let $y = \frac{1}{x}$. $\lim_{y\to -\infty} \arctan(y) = \boxed{-\frac{\pi}{2}}$

c. $\lim_{x\to\infty} f(x)$

As $x \to \infty$, $\frac{1}{x} \to 0^+$. $\arctan(0) = \boxed{0}$

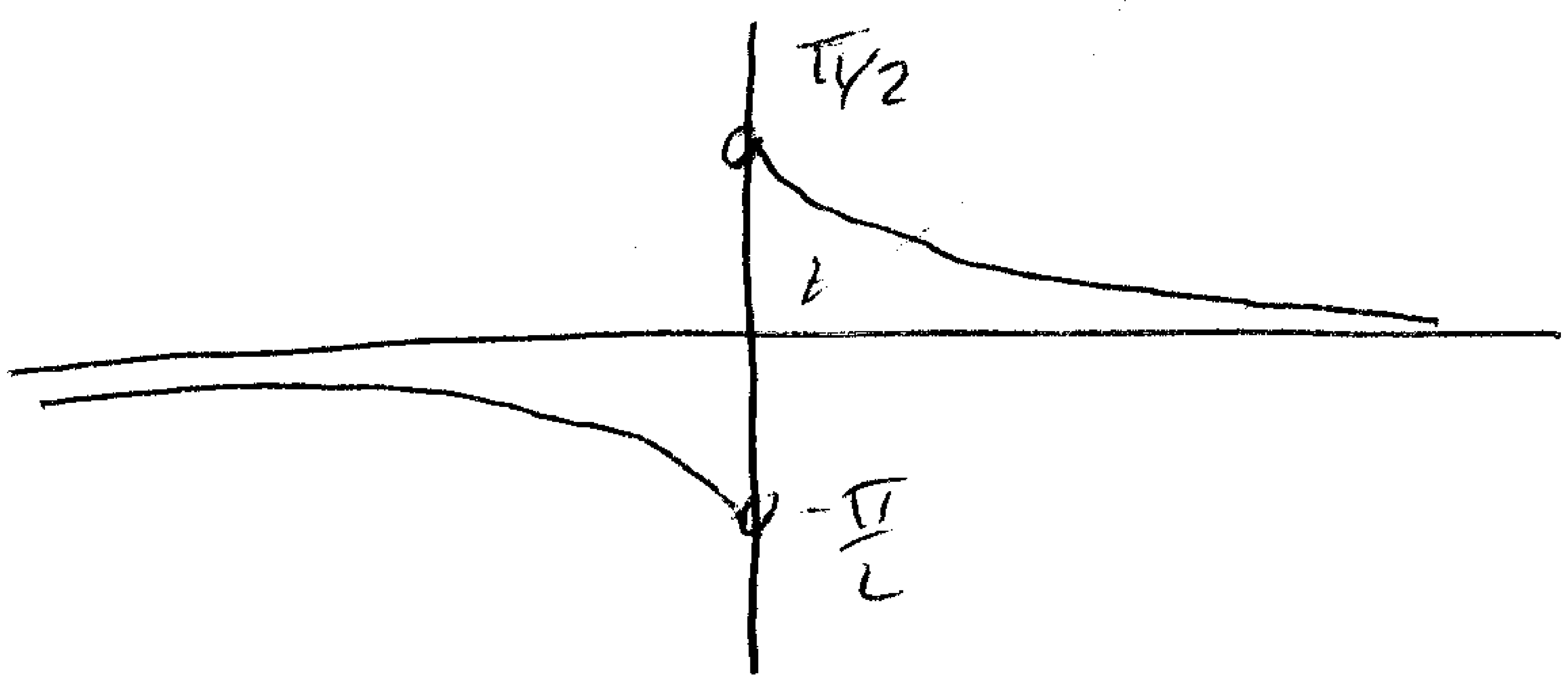
d. $\lim_{x\to-\infty} f(x)$

As $x \to -\infty$, $\frac{1}{x} \to 0^-$. $\arctan(0) = \boxed{0}$

e. Obviously $y = f(x)$ is discontinuous at $x = 0$. What type of discontinuity is this?

Jump discontinuity.

f. Graph $y = f(x)$. Label any asymptotes.



8. [10 points] Let $xy^2 - 2\sin(xy) = 4$. Find y' as a function of x and y.

Here y' means $\frac{dy}{dx}$. Apply $\frac{d}{dx}$ to both sides.

$$(xy^2)' - (2\sin(xy))' = y'$$

$$x'y^2 + x(y^2)' - 2\cos(xy)(xy)' = 0$$

$$y^2 + x \cdot 2yy' - 2\cos(xy)(\underset{1}{x'}y + xy') = 0$$

$$y^2 + 2xyy' - 2\cos(xy)y - 2x\cos(xy)y' = 0$$

$$[2xy - 2x\cos(xy)]y' = -y^2 + 2\cos(xy)$$

$$\boxed{y' = \frac{-y^2 + 2\cos(xy)}{2xy - 2x\cos(xy)}}$$

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1. [5 points] What is the domain of $\frac{x}{\sqrt{5-x}}$? Express your answer in interval notation.

We need $5 - x > 0$. So, $5 > x$ or $(-\infty, 5)$

$x < 5$

2. [5 points] Find the exact value of $\csc^{-1} 2$. Express your answer in radians.

Let $\theta = \csc^{-1} 2$.

Then $\csc \theta = 2$.

So, $\sin \theta = \frac{1}{2}$. Thus $\theta = \boxed{\frac{\pi}{6}}$.

3. [10 points] Let $f(x) = x^2 - \sqrt{x}$. Find the equation in slope-intercept form of the line tangent to the graph of $y = f(x)$ at the point (4,14).

$f'(x) = 2x - \frac{1}{2\sqrt{x}}$. $f'(4) = 8 - \frac{1}{4} = \frac{31}{4}$. This is the slope.

The line through (a,b) with slope m is given by

$$y - b = m(x - a).$$

We have $y - 14 = \frac{31}{4}(x - 4)$.

$$y = \frac{31}{4}x - 31 + 14$$

$$\boxed{y = \frac{31}{4}x - 17}$$

4. [20 points] Find the following limits. Show the steps you are using. Do not just plug in numbers.

$\frac{0}{0}$

a. $\lim_{x\to 0} \frac{\sin 3x}{\sin 4x} = \lim_{x\to 0} \frac{\sin 3x}{1} \cdot \frac{1}{\sin 4x}$

$= \lim_{x\to 0} \frac{3}{4} \frac{\sin 3x}{3x} \frac{4x}{\sin 4x} =$

$\frac{3}{4} \lim_{x\to 0} \left(\frac{\sin 3x}{3x}\right) \cdot \lim_{x\to 0} \left(\frac{4x}{\sin 4x}\right) = \frac{3}{4} \cdot 1 \cdot \frac{1}{1} = \boxed{\frac{3}{4}}$

"$\infty - \infty$"

b. $\lim_{x\to\infty} \frac{\sqrt{9x^2+x} - 3x}{1} \cdot \frac{\sqrt{9x^2+x} + 3x}{\sqrt{} + 3x}$

$= \lim_{x\to\infty} \frac{9x^2+x-9x^2}{\sqrt{} + 3x} = \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x} + 3x}$

$= \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x} + 3x} \cdot \frac{1/x}{1/x} =$

$\lim_{x\to\infty} \frac{1}{\sqrt{9+\frac{1}{x}} + 3} = \frac{1}{\sqrt{9+0}+3} = \boxed{\frac{1}{6}}$

"$\frac{0}{0+0}$"

c. $\lim_{x\to 0} \frac{x}{x+\tan x} \cdot \frac{1/x}{1/x} =$

$\lim_{x\to 0} \frac{1}{1 + \frac{\tan x}{x}} = \frac{1}{1+1} = \frac{1}{2}$

since $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = 1 \cdot 1 = 1$

"$\frac{0}{0}$"

d. $\lim_{x\to 3} \frac{x^2+x-12}{x^2-2x-3} = \lim_{x\to 3} \frac{(x+4)(x-3)}{(x+1)(x-3)}$

$= \lim_{x\to 3} \frac{x+4}{x+1} = \frac{7}{4}$

5. [5 points] Let $f(x)$ and $g(x)$ be differentiable functions. Prove that $(f(x)+g(x))' = f'(x)+g'(x)$.

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$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} = f'(x)+g'(x)$ [redacted]

the limits exist, since $f(x)$ and $g(x)$ are differentiable

6. [30 points] Find the following derivatives.

a. $(x^2 \sin x)' =$

$(x^2)' \sin x + x^2 (\sin x)' =$

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b. $(\sin^3 x)' = 3 \sin^2 x (\sin x)'$

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$= \dfrac{7x+21-7x-1}{(\ )^2} = \dfrac{20}{(x+3)^2}$

d. $(\sqrt{x} \cdot \cos x)' = (\sqrt{x})' \cos x + \sqrt{x} (\cos x)'$

$= \dfrac{1}{2\sqrt{x}} \cos x + \sqrt{x}(-\sin x)$

$= \dfrac{\cos x}{2\sqrt{x}} - \sqrt{x} \cdot \sin x$

e. $(\cot 3x)' =$

$-\csc^2(3x) \cdot (3x)'$

$= -3 \csc^2(3x)$

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b. $\lim_{x\to 0^-} f(x)$ As $x \to 0^-$, $\frac{1}{x} \to -\infty$. Let $y = \frac{1}{x}$. $\lim_{y\to -\infty} \arctan(y) = \boxed{-\frac{\pi}{2}}$

c. $\lim_{x\to\infty} f(x)$ As $x \to \infty$, $\frac{1}{x} \to 0$. $\arctan(0) = \boxed{0}$

d. $\lim_{x\to -\infty} f(x)$ As $x \to -\infty$, $\frac{1}{x} \to 0$. $\arctan(0) = \boxed{0}$

e. Obviously $y = f(x)$ is discontinuous at $x = 0$. What type of discontinuity is this?

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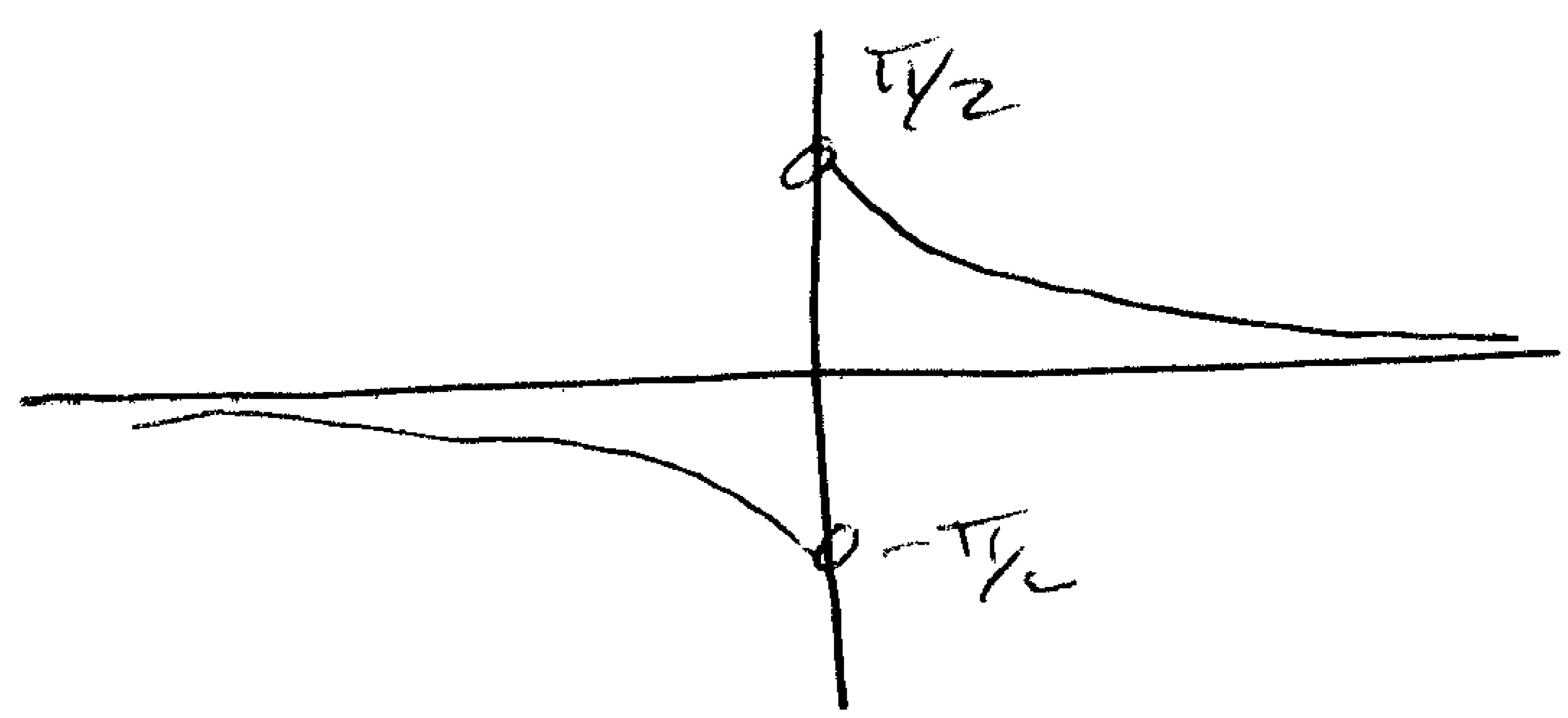

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$$(xy^2)' - (2\sin(xy))' = 4'$$

$$x'y^2 + x(y^2)' - 2\cos(xy)\cdot(xy)' = 0$$

$$y^2 + x\cdot 2yy' - 2\cos(xy)(\underset{1}{x'}y + xy') = 0$$

$$y^2 + 2xyy' - 2y\cos(xy) - 2x\cos(xy)y' = 0$$