

Name: Key

NON-GRAPHING CALCULATORS ALLOWED

1. [5 points each] Compute each of the following derivatives. You do not need to simplify your answers.

a. $(\ln(x + \sin x))'$

$$\frac{1 + \cos x}{x + \sin x}$$

b. $((x^2 + \tan^{-1} 2x + 3^x)^{1/3})'$

$$\frac{1}{3} (x^2 + \tan^{-1}(2x) + 3^x)^{-2/3} \cdot \left(2x + \frac{2}{1+4x^2} + 3^x \ln 3 \right)$$

c. $\left(\frac{x - \csc x}{x^2 + 3} \right)'$

$$\frac{(1 + \csc x \cot x)(x^2 + 3) - (x - \csc x)(2x)}{(x^2 + 3)^2}$$

d. $(xe^{\sinh 2x})'$

$$e^{\sinh x} + x \cosh x e^{\sinh x}$$

e. $\left[\log_4(\sin^3(4x^6)) \right]'$

$$\frac{1}{\ln 4 \sin^3(4x^6)} \cdot (\sin^3(4x^6))'$$

$$= \frac{3 \sin^2(4x^6)}{\ln 4 \sin^3(4x^6)} (\sin(4x^6))'$$

$$= \frac{3}{\ln 4 \sin(4x^6)} \cos(4x^6) \cdot 24x^5$$

$$= \frac{72}{\ln 4} \cot(4x^6)$$

f. Let $y = (\ln x)^{x+1}$. Find $\frac{dy}{dx}$.

$$\ln y = \ln ((\ln x)^{x+1})$$

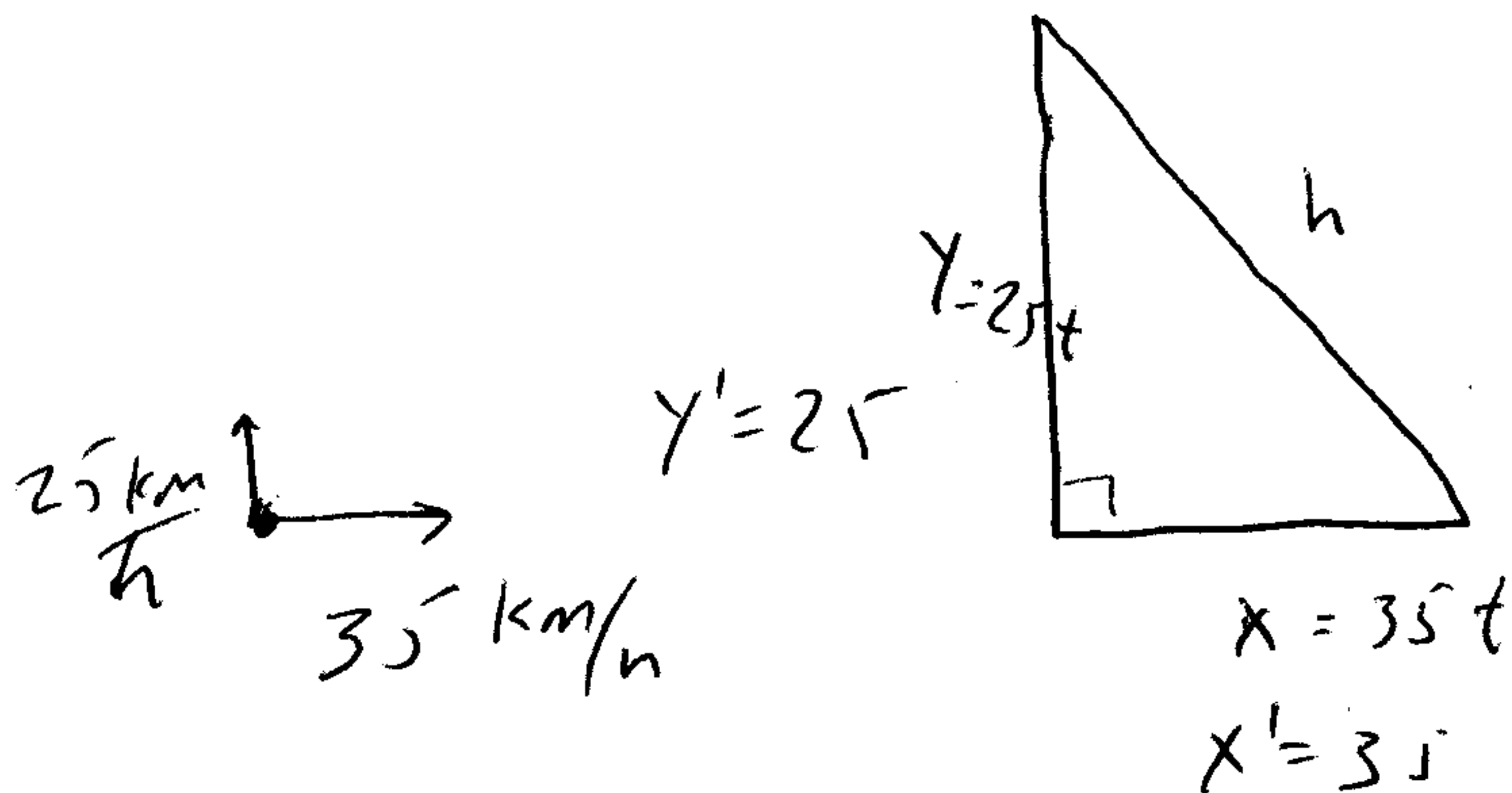
$$\ln y = (x+1) \ln(\ln x)$$

$$\frac{y'}{y} = \ln(\ln x) + (x+1) \frac{1}{\ln x} (\ln x)'$$

$$y' = y \left[\ln \ln x + \frac{x+1}{x \ln x} \right]$$

$$= (\ln x)^{x+1} \left[\ln \ln x + \frac{x+1}{x \ln x} \right]$$

2. [10 points] Ship A and Ship B leave the same port at noon. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 p.m.?



$$h^2 = x^2 + y^2$$

$$2hh' = 2xx' + 2yy'$$

$$h' = \frac{xx' + yy'}{h}$$

at $t = 4$ hours, $x = 35 \cdot 4 = 140$

$y = 25 \cdot 4 = 100$

Therefore at $t = 4$ $h = \sqrt{(140)^2 + (100)^2} \approx 172.05$ [STORE in memory]

$$h' = \frac{140 \cdot 35 + 100 \cdot 25}{172.05} \approx \boxed{43.5 \frac{\text{km}}{\text{h}}}$$

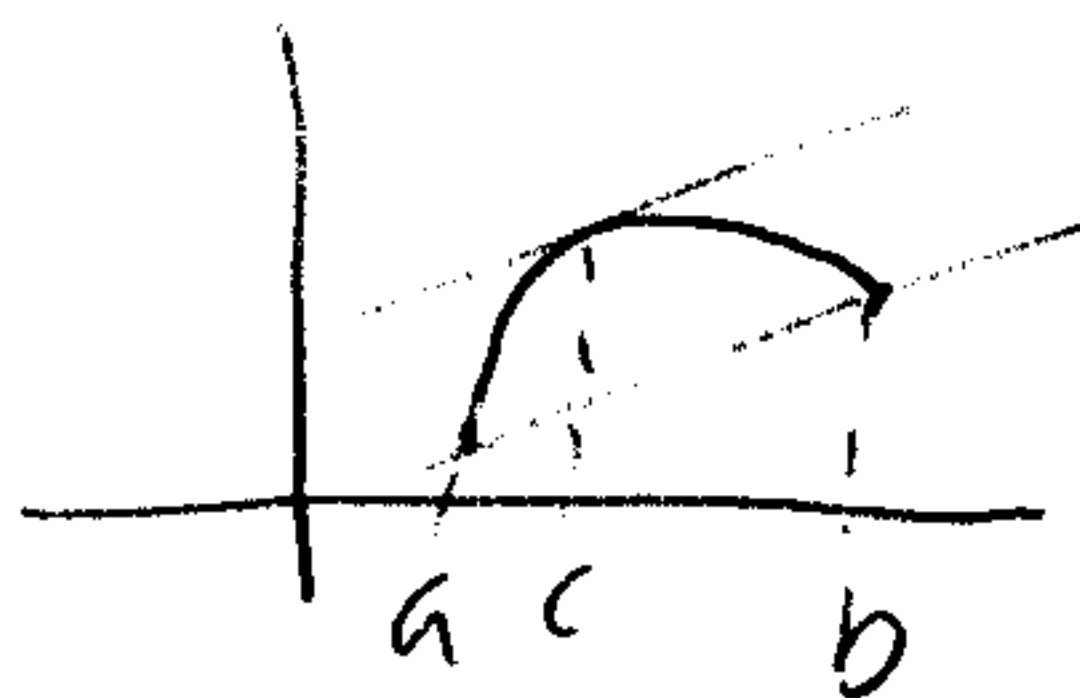
3. [20 points] a. State the Mean Value Theorem by filling in the blank spaces below.

The Mean Value Theorem: Let $[a, b]$ be a closed bounded interval. Suppose f is a function whose domain contains $[a, b]$ that satisfies the following two conditions.

1. f is continuous on $[a, b]$.
2. f is differentiable on (a, b) .

Then there exists a number $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

- b. Draw a picture that illustrates the idea behind the Mean Value Theorem.



- c. Suppose the $f(0) = -3$ and that $f'(x) \leq 5$ for all values of x . How large can $f(2)$ possibly be? Justify your answer.

on $[0, 2]$ f is cont. since $f'(x)$ exists.
and f is diff. on $(0, 2)$. For some $c \in (0, 2)$

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(0)}{2 - 0} = \frac{f(2) + 3}{2}$$

Since $f'(c) \leq 5$, we have $\frac{f(2) + 3}{2} \leq 5$. Hence $f(2) \leq 10 - 3 = 7$.

Thus 7 is the largest f can be.

- d. Show that the equation $3x - 1 - \cos x = 0$ has exactly one real solution. Hint: first show there is a solution in the interval $(0, \pi)$.

$$\text{Let } f(x) = 3x - 1 - \cos x. \quad f(0) = 0 - 1 - 1 = -2$$

$$f(\pi) = 3\pi - 1 - (-1) = 3\pi. \quad \text{Since } -2 < 0 < 3\pi$$

the Intermediate Value Thm says for some $c \in (0, \pi)$

$$f(c) = 0. \quad \text{In order for } f(x) \text{ to have another zero}$$

there must be a place (c) where $f'(c) = 0$. This is by Rolle's

Theorem. However $f'(x) = 3 + \sin x$. This can never be

zero since $\sin x$ is never $= -3$.

Thus, $f(x)$ cannot have a second zero.

4. [10 points] Sketch the graph of a function $y = f(x)$ that has the following properties. You may assume f is continuous and has continuous first and second derivatives.

$$f(0) = 3$$

$$\text{For } x \text{ in } (-\infty, -2) \quad f'(x) > 0.$$

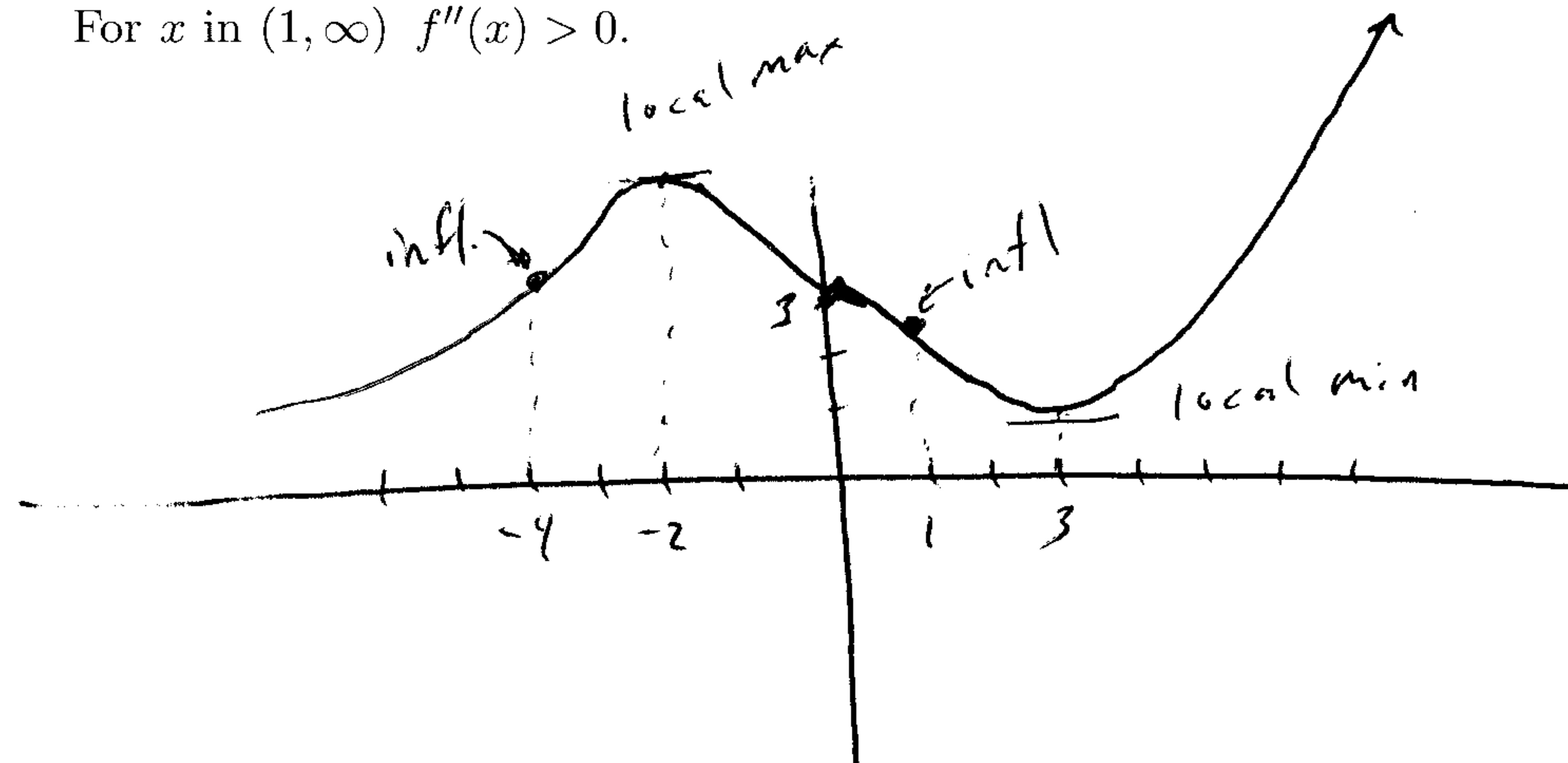
$$\text{For } x \text{ in } (-2, 3) \quad f'(x) < 0.$$

$$\text{For } x \text{ in } (3, \infty) \quad f'(x) > 0.$$

$$\text{For } x \text{ in } (-\infty, -4) \quad f''(x) > 0.$$

$$\text{For } x \text{ in } (-4, 1) \quad f''(x) < 0.$$

$$\text{For } x \text{ in } (1, \infty) \quad f''(x) > 0.$$



If you said $(-2, -4)$ that is ok.

5. [10 points] Let $f(x)$ be a function. Below we show the **graph of the derivative**, $y = f'(x)$. Based on this graph answer these questions about the original function $f(x)$.

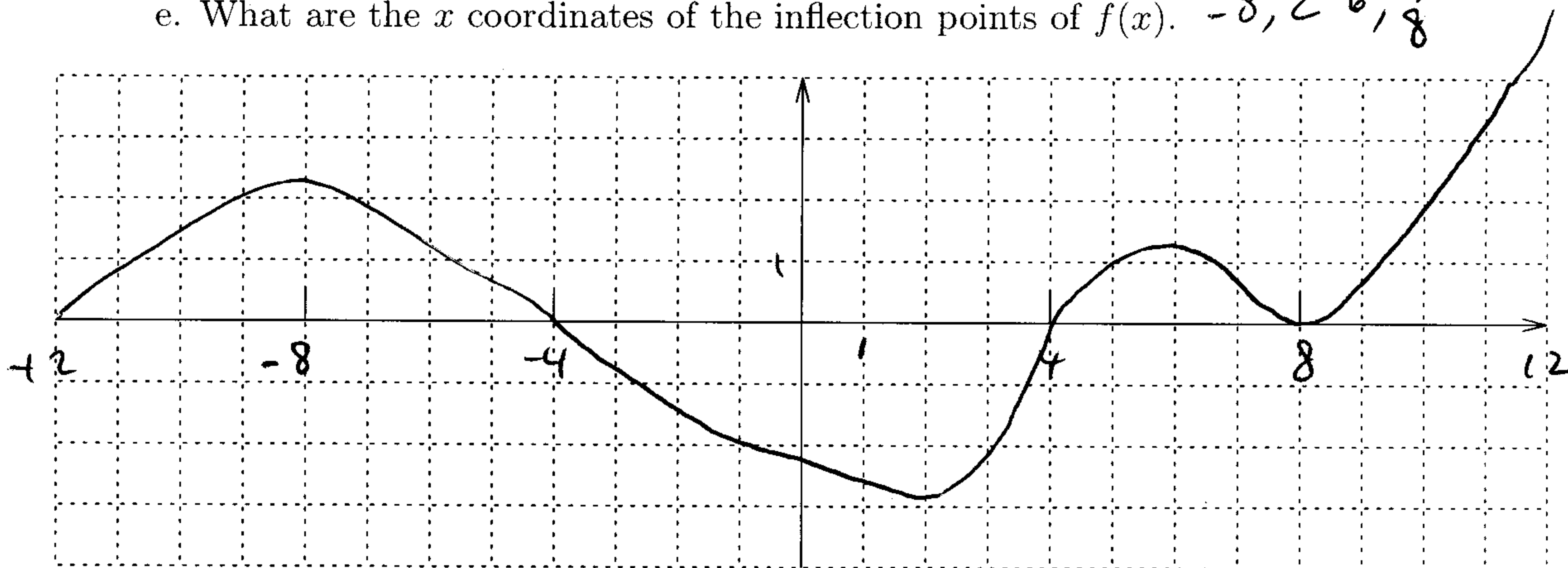
a. On what intervals is $f(x)$ increasing? $(-12, -4), (4, 12)$

b. On what intervals is $f(x)$ decreasing? $(-4, 4)$

c. On what intervals is $f(x)$ concave up? $(-12, -8), (2, 6), (8, 12)$

d. On what intervals is $f(x)$ concave down? $(-8, 2), (6, 8)$

e. What are the x coordinates of the inflection points of $f(x)$. $-8, 2, 6, 8$



6. [10 points] Sketch the graph of $y = e^{-x^2}$. Find the x coordinates of the two inflection points. Indicate where the graph is concave up and concave down.

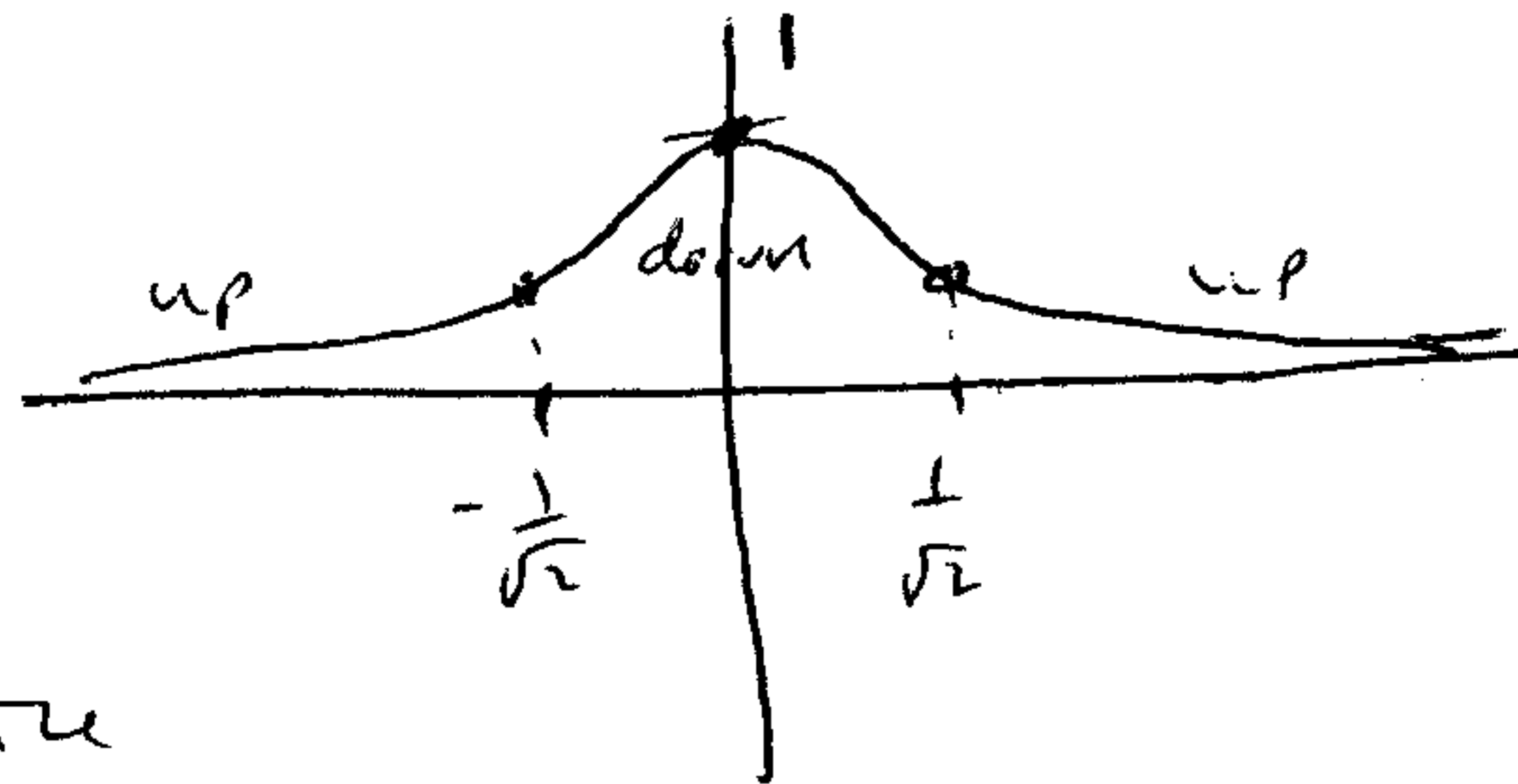
$$y' = -2x e^{-x^2}$$

$$y'' = -2e^{-x^2} + 4x^2 e^{-x^2}$$

$$= (4x^2 - 2) e^{-x^2}$$

$$y'' = 0 \text{ when } x = \pm \frac{1}{\sqrt{2}} \approx \pm 0.707$$

These are the locations of the two infli. pts.



7. [10 points] Sketch the graph of $y = \frac{x}{x^2 + 1}$. Find the values and locations of the absolute maximum and absolute minimum.

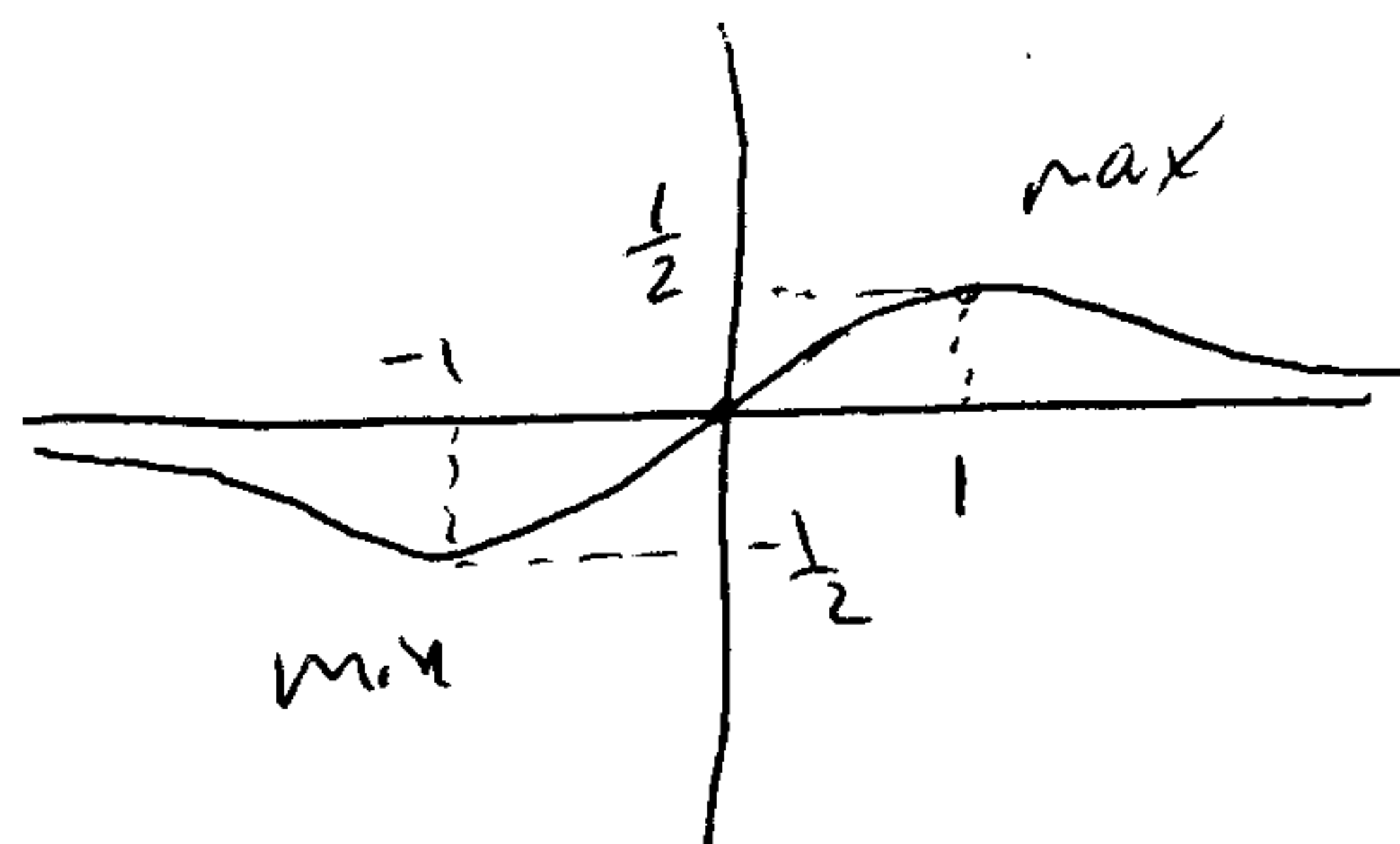
$$y' = \frac{1(x^2 + 1) - (x)(2x)}{(x^2 + 1)^2}$$

$$= \frac{1 - x^2}{(x^2 + 1)^2} = 0$$

$$\text{when } x = \pm 1$$

$$y(1) = \frac{1}{1^2 + 1} = \frac{1}{2}$$

$$y(-1) = -\frac{1}{2}$$



8. [10 BONUS points] Let $f(x)$ be a twice differential function that has a relative maximum at $x = c$ and has negative second derivative at $x = c$. Prove that $\arctan(f(x))$ must also have a relative maximum at $x = c$.

$$(\arctan(f(x)))' = \frac{1}{1+f^2} f' \quad \text{It is 0 at } x=c \text{ since } f'(c) = 0.$$

$$(\arctan(f(x)))'' = \frac{f''(1+f^2) - f'(2ff')}{(1+f^2)^2} \xrightarrow{\text{at } x=c} = \frac{f''}{1+f^2} < 0$$

Since $f''(c) < 0$.

Thus $\arctan(f(x))$ has a rel max at $x=c$.