LIMIT DEFINITIONS

These are the formal definitions of each type of limit. You do not need to memorize these. Under each is the corresponding informal definition. You might look them over and see why they make sense. Recall $\forall$ mean "for every" and $\exists$ means "there exists".

1. $\lim_{x\to a} f(x) = L$ means $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.
    - As x approaches a from either side $f(x)$ approaches L.
2. $\lim_{x\to a^+} f(x) = L$ means $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that $0 < x - a < \delta$ implies $|f(x) - L| < \epsilon$.
    - As x approaches a from the right side $f(x)$ approaches L.
3. $\lim_{x\to a^-} f(x) = L$ means $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that $0 < a - x < \delta$ implies $|f(x) - L| < \epsilon$.
    - As x approaches a from the left side $f(x)$ approaches L.
4. $\lim_{x\to \infty} f(x) = L$ means $\forall\, \epsilon > 0\ \exists\, M > 0$ such that $x > M$ implies $|f(x) - L| < \epsilon$.
    - As x grows positively without bound $f(x)$ approaches L.
5. $\lim_{x\to -\infty} f(x) = L$ means $\forall\, \epsilon > 0\ \exists\, M < 0$ such that $x < M$ implies $|f(x) - L| < \epsilon$.
    - As x grows negatively without bound $f(x)$ approaches L.
6. $\lim_{x\to a} f(x) = \infty$ means $\forall\, B > 0\ \exists\, \delta > 0$ such that $0 < |x - a| < \delta$ implies $f(x) > B$.
    - As x approaches a from either side $f(x)$ grows positively without bound. (Up, up, and Away!)
7. $\lim_{x\to a^+} f(x) = \infty$ means $\forall\, B > 0\ \exists\, \delta > 0$ such that $0 < x - a < \delta$ implies $f(x) > B$.
    - As x approaches a from the right side $f(x)$ grows positively without bound.
8. $\lim_{x\to a^-} f(x) = \infty$ means $\forall\, B > 0\ \exists\, \delta > 0$ such that $0 < a - x < \delta$ implies $f(x) > B$.
    - As x approaches a from the left side $f(x)$ grows positively without bound.
9. $\lim_{x\to a} f(x) = -\infty$ means $\forall\, B < 0\ \exists\, \delta > 0$ such that $0 < |x - a| < \delta$ implies $f(x) < B$.
    - As x approaches a from either side $f(x)$ grows negatively without bound.

10. $\lim_{x \to a^+} f(x) = -\infty$ means $\forall\, B < 0\ \exists\, \delta > 0$ such that $0 < x - a < \delta$ implies $f(x) < B$.
    - As x approaches a from the right side $f(x)$ grows negatively without bound.

11. $\lim_{x \to a^-} f(x) = -\infty$ means $\forall\, B < 0\ \exists\, \delta > 0$ such that $0 < a - x < \delta$ implies $f(x) < B$.
    - As x approaches a from the left side $f(x)$ grows negatively without bound.

12. $\lim_{x \to \infty} f(x) = \infty$ means $\forall\, B > 0\ \exists\, M > 0$ such that $x > M$ implies $f(x) > B$.
    - As x grows positively without bound $f(x)$ grows positively without bound.

13. $\lim_{x \to \infty} f(x) = -\infty$ means $\forall\, B < 0\ \exists\, M > 0$ such that $x > M$ implies $f(x) < B$.
    - As x grows positively without bound $f(x)$ grows negatively without bound.

14. $\lim_{x \to -\infty} f(x) = \infty$ means $\forall\, B > 0\ \exists\, M < 0$ such that $x < M$ implies $f(x) > B$.
    - As x grows negatively without bound $f(x)$ grows positively without bound.

15. $\lim_{x \to -\infty} f(x) = -\infty$ means $\forall\, B < 0\ \exists\, M < 0$ such that $x < M$ implies $f(x) < B$.
    - As x grows negatively without bound $f(x)$ grows negatively without bound.

Optional Exercises.

1. Draw a function that satisfies 1 for $a = 2$ and $L = 3$, 2 for $a = -1$ and $L = 7$, 3 for $a = -1$ and $L = 4$, 13 and 15.

2. Draw a function that satisfies 11 for $a = 1$, 2 for $a = 1$ and $L = 4$, 6 for $a = 4$, 4 for $L = -6$ and 5 for $L = 0$.