Math 150 Test 2 Fall 2011

Name: Key

NON-GRAPHING CALCULATORS ALLOWED

1. [20 points] Find the derivatives of the following functions.

a. $(\sin x)^x$

$y = (\sin x)^x$

$\ln y = \ln((\sin x)^x) = x \ln(\sin x)$

$\frac{y'}{y} = \ln(\sin x) + \frac{x}{\sin x} \cdot \cos x$

$y' = (\sin x)^x \left[\ln(\sin x) + x \cot x\right]$

b. $\left((\ln x)(\tan^{-1}(\sqrt{x}))\right)' =$

$\frac{1}{x} \tan^{-1}(\sqrt{x}) + \ln x \cdot \frac{1}{1+(\sqrt{x})^2} (\sqrt{x})'$

$= \frac{\tan^{-1}(\sqrt{x})}{x} + \frac{\ln x}{(1+x)2\sqrt{x}}$

c. $\left(\sinh^3(7^x)\right)'$

$3\sinh^2(7^x)\cosh(7^x) \cdot (7^x)'$

$= 3\sinh^2(7^x)\cosh(7^x) \cdot 7^x \ln 7$

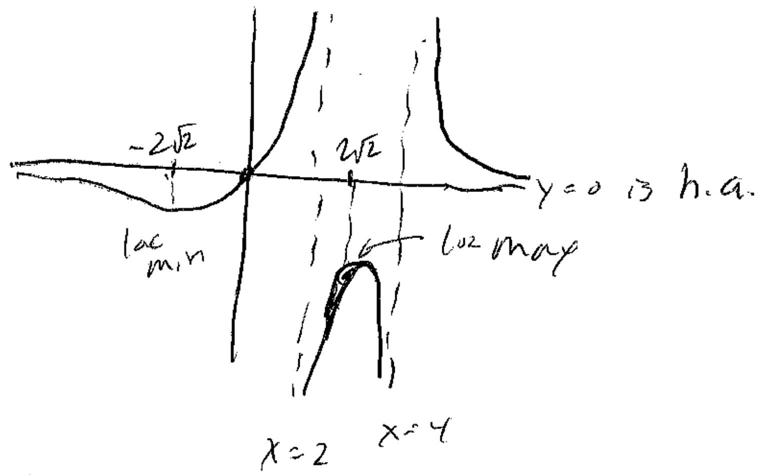
c. $\left(\cot^3(7e^x)\right)' =$

$3\cot^2(7e^x)\left(-\csc^2(7e^x)\right) \cdot (7e^x)'$

$= -21 e^x \cot^2(7e^x)\csc^2(7e^x)$

2. [10 points] Let $f(x) = \frac{x}{x^2-6x+8}$. Sketch the graph of $y = f(x)$. Label the vertical asymptotes and indicate the horizontal asymptote. Find the x coordinates of the two local extrema and mark where they are on your graph. You'll be able to see which one is a local minimum and which one is a local maximum so you needn't bother with the Second Derivative test.

$$f(x) = \frac{x}{(x-2)(x-4)}$$



$$f' = \frac{(x^2-6x+8) - x(2x-6)}{(\quad)^2}$$

$$= \frac{-x^2+8}{(\quad)^2}$$

$f'(x) = 0$ at $x = \pm\sqrt{8} = \pm 2\sqrt{2} \approx \pm 2.828$

3. [10 points] Let $f(x) = 4x^5 + x^3 + 2x + 1$. Show that is has a real root between $x = -1$ and $x = 0$. Show that is has no other real roots.

$f(-1) = -4 - 1 - 2 + 1 = -6 < 0$

$f(0) = 1 > 0$

By the IVT there at least one number $c \in (-1, 0)$ for which $f(c) = 0$. If there was another, any where on the real line, by Rolle's Thm there would be a number between them where $f'(x) = 0$. But

$f'(x) = 20x^4 + 3x^2 + 2$ is clearly alway ≥ 2.

Hence there is only one plase where $f(x) = 0$.

4. [10 points] For which values of c will the function $f(x) = cx + \sin x$ have no local extrema?

$f'(x) = c + \cos x$. For $c > 1$ or $c < -1$,

$f'(x)$ will not have any critical numbers.

. Thus $f(x)$ will have no local extrema.

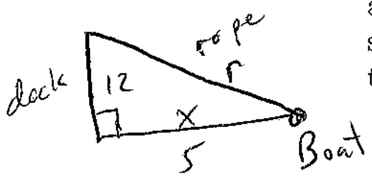
If $c \in (-1, 1)$ there there will be critical numbers.

Now $f''(x) = -\sin x$. So, $f''(c)$ can only be zero if $c = n\pi$. This won't happen for $c \in (-1, 1)$.

It will happen for $c = \pm 1$. In this case the c.n. are inflection pts and there are no local extrema.

So $f''(c) \neq 0$ means loc. min or max.

5. [10 points] A boat is pulled in by means of a winch on a dock 12 feet above the deck of the boat. If the boat is approaching the dock at a speed of 2 feet per second, how fast is the rope being pulled in when the boat is 5 feet from the dock?

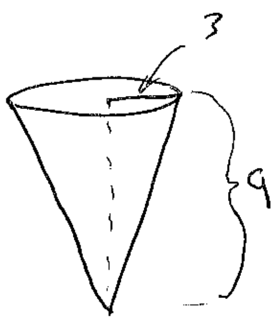


$r^2 = x^2 + 12^2 = x^2 + 144$

$2rr' = 2xx'$

$r' = \frac{x\,x'}{r} = \frac{5(-2)}{\sqrt{25+144}} = \frac{-10}{\sqrt{169}} = \frac{-10}{13} \approx -0.7692$ ft/sec

Thus the rope is being pulled in
0.7692 ft/sec



6. [10 points] A water tank has the shape of an inverted circular cone with the radius of the top equal to 3 meters and a maximum depth of 9 meters. If water is being pumped in at a rate of 5 m^3/min, what is the rate of change of the water level when the water is 6 meters deep? Give your answer is m/min rounded to the nearest 3rd decimal place.

$V = \frac{1}{3}\pi r^2 h$ $h = 3r$

$r = h/3$

$V = \frac{1}{3}\pi \frac{h^3}{9} = \frac{\pi h^3}{27}$

$\frac{dV}{dt} = \frac{\pi h^2}{9}\frac{dh}{dt}$

$5 = \frac{\pi 6^2}{9}\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{45}{36\pi} = 0.398 \frac{m}{min}$

7. [15 points] Let $f(x) = e^{-x} \cdot \ln x$ for $x > 0$.
a. Find $f'(x)$ and show that it must be zero somewhere between $x = 1$ and $x = e$.
b. Find $f''(x)$. Simplify your answer.
c. Suppose we have determined numerically that $f'(1.763222834) = 0$. Use the Second Derivative Test to determine whether it is a local minimum or maximum.

$f'(x) = -e^{-x} \ln x + \frac{e^{-x}}{x} = e^{-x}\left(\frac{1}{x} - \ln x\right).$

$f'(1) = \frac{1}{e}(1-0) = \frac{1}{e} > 0 \qquad f'(e) = \frac{1}{e^e}\left(\underbrace{\frac{1}{e}}_{<1} - 1\right) < 0.$

Thus by IVT $\exists\, c \in (1,e)$ s.t. $f'(c) = 0$.

$f''(x) = -e^{-x}\left(\frac{1}{x} - \ln x\right) + e^{-x}\left(\frac{-1}{x^2} - \frac{1}{x}\right) = -e^{-x}\left(\frac{2}{x} + \frac{1}{x^2} - \ln x\right)$

$f''(1.76\ldots) = -(0.1714\ldots)(1.13\ldots + 0.321\ldots + 0.567) \approx -3.567.$

Thus $x = 1.76\ldots$ is a local max.

8. [5 points] Show that $y = \ln x$ is increasing on $(0, \infty)$ but that

$$y = \frac{2}{x} + \frac{1}{x^2}$$

is decreasing on $(0, \infty)$. Hence they can cross at most once. Show that they the do cross. What does this tell you about the number of inflection points in the previous problem? Explain.

$(\ln x)' = \frac{1}{x} > 0$ on $(0,\infty)$ and is therefore increasing.

$\left(\frac{2}{x} + \frac{1}{x^2}\right)' = -\frac{2}{x^2} - \frac{2}{x^3} < 0$ on $(0,\infty)$ and is therefore decreasing.

$\ln(1) = 0 \quad \ln(e) = 1$

$\left(\frac{2}{1} + \frac{1}{1^2}\right) = 3$ and $\frac{2}{e} + \frac{1}{e^2} \approx 0.87 < 1$. So they crossed.

Thus $f''(x)$ in #7 ~~[illegible]~~ is zero exactly once and f has one infl. pt.

8. [10 points] Sketch the graph of a function $f(x)$ that satisfies the following conditions.

$f'(0) = f'(1) = f'(5) = 0.$

$f'(x) > 0$ if $x < 0$ or $1 < x < 5.$

$f'(x) < 0$ if $0 < x < 1$ or $5 < x.$

$f''(x) > 0$ if $1/2 < x < 3$ or $x < -3.$

$f''(x) < 0$ if $-3 < x < 1/2$ or $3 < x.$