Math 150 Test 2 Fall 2012

Name: Key Section time: ______________

SCIENTIFIC CALCULATORS ALLOWED

1. [5 points] Find $\lim_{x\to\infty} \ln(2x^3+1) - 3\ln(x+1)$. Hint, combine into a single ln function.

$= \ln \frac{2x^3+1}{(x+1)^3}$

$\lim_{x\to\infty} \ln\left(\frac{2x^3+1}{x^3+3x^2+3x+1}\right) = \ln\left(\lim_{x\to\infty} \frac{2x^3+1}{x^3+\cdots 1}\right) = \boxed{\ln 2}$

2. [5 points] Find $\lim_{x\to\infty} \arctan\left(\frac{7x^2+3x-1}{7x^2+6x+4}\right)$. $=$ ~~tan~~ $\arctan\left(\frac{7}{7}\right) = \arctan(1)$ $\boxed{= \frac{\pi}{4}}$

3. [5 points] [Calculator required] Find all values of x such that

$$\ln(x+5) + \ln(x+2) = 2.302585093$$

$\ln (x+5)(x+2) =$ "

$(x+5)(x+2) = e^{2.3\ldots} = 10$

$x^2+7x+10 = 10$

$x^2+7x = 0$

$x(x+7) = 0$

$x = 0$ or $x = -7$.

But -7 is not the ~~do~~ domain of $\ln x+5 + \ln x+2$.

So the only solution is $\boxed{x = 0.}$

4. [20 points] Find the derivatives of the functions below.

a. x^{7x}

$y = x^{7x}$

$\ln y = 7x \ln x$

$\frac{y'}{y} = 7\ln x + 7\frac{x}{x}$

$y' = 7x^{7x}(1+\ln x)$

b. $\tan^{-1}(\sec x)$

$\frac{1}{1+\sec^2 x}(\sec x)' =$

$\frac{\sec x \tan x}{1+\sec^2 x}$

c. $e^{\cos x} + \ln x^3$

$-\sin x\, e^{\cos x} + \frac{1}{x^3}\cdot 3x^2$

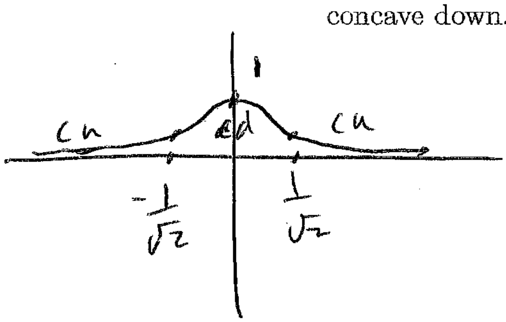
$= -\sin x\, e^{\cos x} + \frac{3}{x}$

(or use $\ln x^3 = 3\ln x$)

d. $\cot(5x + e^x)$

$-\csc^2(5x+e^x)\cdot(5+e^x)$

5. [10 points] Sketch the graph of $y = e^{-x^2}$. Find the x coordinates of the two inflection points. Indicate where the graph is concave up and concave down.



$y' = -2x e^{-x^2}$

$y'' = -2e^{-x^2} + 4x^2 e^{-x^2}$

$= 2e^{-x^2}(2x^2-1) \overset{\text{set}}{=} 0$

$2x^2 = 1$

$x^2 = \pm\frac{1}{\sqrt{2}}$

6. [10 points] Let

$$f(x) = \frac{x}{x^2 - 6x + 8}. \quad = \frac{x}{(x-4)(x-2)}$$

U.A. $x=4, x=2$
H.A. $y=0$.

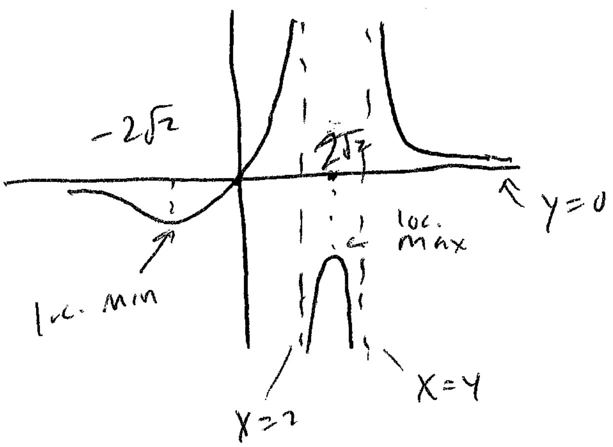
Sketch the graph of $y = f(x)$. Label the vertical asymptotes and indicate the horizontal asymptote. Find the x coordinates of the two local extrema and mark where they are on your graph. You'll be able to see which one is a local minimum and which one is a local maximum so you needn't bother with the Second Derivative Test.

$$f'(x) = \frac{(x^2-6x+8) - x(2x-6)}{(\quad)^2}$$

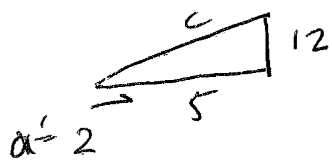
$$= \frac{-x^2+8}{(\quad)^2} \overset{set}{=} 0$$

$x = \pm\sqrt{8} = \pm 2\sqrt{2}$

$2\sqrt{2}$ gives loc max
$-2\sqrt{2}$ gives loc. min



7. [10 points] A boat is pulled in by means of a winch on a dock 12 feet above the deck of the boat. If the boat is approaching the dock at a speed of 2 feet per second, how fast is the rope being pulled in when the boat is 5 feet from the dock?



$a^2 + 12^2 = c^2$

$2aa' = 2cc'$

$c' = \frac{aa'}{c}$ $\quad c = \sqrt{12^2+5^2} = 13$

$= \frac{5\cdot(-2)}{13} = -\frac{10}{13}.$

Rope is being pulled in by $\frac{10}{13}$ ft/sec.

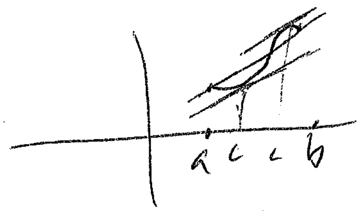
8. [20 points] a. State the Mean Value Theorem by filling in the blank spaces below.

The Mean Value Theorem: Let $[a, b]$ be a closed bounded interval. Suppose f is a function whose domain contains $[a, b]$ that satisfies the following two conditions.

1. f is continuous on $[a, b]$.
2. f is differentiable on (a, b).

Then there exists a number $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

b. Draw a picture that illustrates the idea behind the Mean Value Theorem.



c. Let $f(x)$ be continuous and differentiable for all real numbers x. Suppose that $f(0) = -3$ and that $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be? Justify your answer.

$$\frac{f(2) - f(0)}{2 - 0} \leq 5$$

$$f(2) + 3 \leq 10$$

$$f(2) \leq 7$$

If $f' \leq 5$ we go ≤ 10 miles in 2 hours. Since we started at -3, $f(2) \leq -3 + 10 = 7$.

d. Let $f(x) = 3x - 1 - \cos x$. Show that $f(x) = 0$ has exactly one real solution. Hint: first show there is a solution in the interval $(0, \pi)$.

$f(0) = -2$ $f(\pi) = 3\pi - 1 + 1 = 3\pi$. Since $-2 < 0 < 3\pi$

$\exists c \in (0, \pi)$ s.t. $f(c) = 0$. Suppose there is a second solution. Call it d. Then by MVT

$$\frac{f(d) - f(c)}{d - c} = f'(g) \text{ for some } g \in (c, d).$$

Since $f(d) = f(c) = 0$ we get $f'(g) = 0$.

But since $f'(x) = 3 + \sin x$ can never be zero.

Hence there are no solutions besides c.

9. [10 points] Prove the *Increasing Test.* Specifically prove that if $f'(x) > 0$ on (a, b) then f is increasing on (a, b).

   Start by letting x_1 and x_2 be in (a, b) with $x_1 < x_2$. Use the Mean Value Theorem to show that $f(x_1) < f(x_2)$. Thus f is increasing on (a, b).

Since f is differentiable on (a,b) it is cont. on $[x_1, x_2]$ and diff on (x_1, x_2). So the MVT applies. Thus $\exists\, c \in (x_1, x_2)$ s.t. $\dfrac{f(x_2)-f(x_1)}{x_2 - x_1} = f'(c)$.

Since $f'(c) > 0$ and $x_2 - x_1 > 0$ we have $f(x_2) - f(x_1) > 0$. Thus $f(x_2) > f(x_1)$, which means f is increasing.

10. [5 points] Find the critical numbers of $f(x) = \sin x + \cos x$ in $[0, 2\pi]$. Use the Second Derivative Test to determine if they are relative minimum or maximum locations.

$f'(x) = \cos x - \sin x = 0 \Rightarrow \cos x = \sin x$
$\Rightarrow \tan x = 1$
$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$.

$f''(x) = -\sin x - \cos x$.

$f''(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2} < 0$. Thus $f(\frac{\pi}{4})$ is a local max.

$f''(\frac{5\pi}{4}) = -(-\frac{\sqrt{2}}{2}) - (-\frac{\sqrt{2}}{2}) = \sqrt{2} > 0$. Thus $f(\frac{5\pi}{4})$ is a local min.

Test 2 Bonus Questions

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NAME

Use L'Hôpital's Rule to compute the following limits. (2 points each)

1. $$\lim_{x\to 0} \frac{x^2}{1-\cos x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{2x}{\sin x} = 2\lim_{x\to 0} \frac{x}{\sin x} = 2\cdot 1 = 2.$$

2. $$\lim_{x\to 0} \frac{8^x - 5^x}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{8^x \ln 8 - 5^x \ln 5}{1} = 8^0 \ln 8 - 5^0 \ln 5$$
$$= \ln 8 - \ln 5 = \ln \frac{8}{5} \approx 0.47$$

3. $$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

4. $$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{-\frac{\infty}{\infty}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} (-x) = 0.$$