Name: ______________________________

Math 150 Test 1 Fall 2024

ONLY SCIENTIFIC CALCULATORS ALLOWED

1. [5 points each] Find the following limits. Show your work.

a. $\lim\limits_{x\to 0} \dfrac{\tan 2x}{x}$

$$= \lim_{x\to 0} 2\frac{\sin 2x}{2x}\cdot\frac{1}{\cos 2x}$$

$$= 2\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\lim_{x\to 0}\frac{1}{\cos(2x)}$$

$$= 2\cdot 1\cdot\frac{1}{\cos(0)} = \frac{2}{1} = 2.$$

b. $\lim\limits_{t\to\infty} \dfrac{\cos 3t}{t} = 0$

$$-1 \le \cos 3t \le 1$$

$$-\frac{1}{t} \le \frac{\cos 3t}{t} \le \frac{1}{t}$$

Since $-\frac{1}{t}$ and $\frac{1}{t} \to 0$ as $t\to\infty$, the Squeeze Thm implies $\frac{\cos(3t)}{t} \to 0$ as $t\to\infty$.

Thus, the limit is 0.

c. $\lim\limits_{x\to\infty} \dfrac{2x - \sqrt{4x^2+x}}{1}\cdot\dfrac{2x+\sqrt{}}{2x+\sqrt{}}$

$$= \lim_{x\to\infty}\frac{4x^2-(4x^2+x)}{2x+\sqrt{4x^2+x}}$$

$$= \lim_{x\to\infty}\frac{-x}{2x+\sqrt{4x^2+x}}\ \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to\infty}\frac{-1}{2+\sqrt{4+\frac{1}{x}}}$$

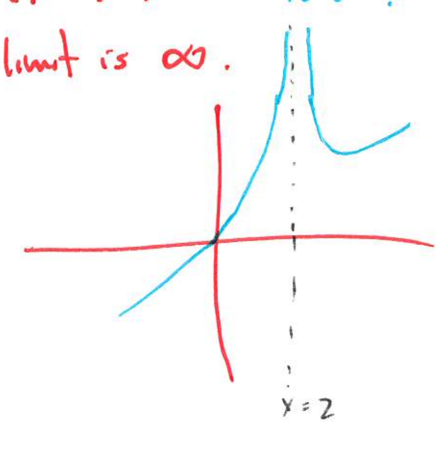
$$= \frac{-1}{2+\sqrt{4+0}} = -\frac{1}{4}$$

d. $\lim\limits_{x\to 2} \dfrac{x^3}{x^2-4x+4}$ (Hint: Factor the denominator.)

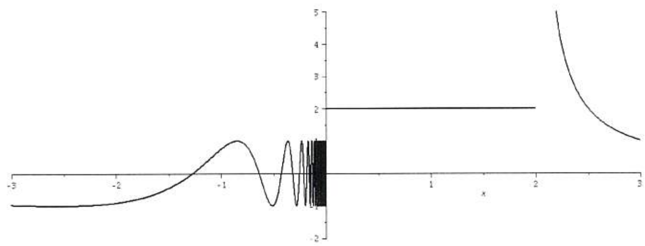
$$= \lim_{x\to 2}\frac{x^3}{(x-2)^2} = \text{“}\frac{8}{0}\text{”}$$

near $x=2$ $x^3 \approx 8 > 0$, while $(x-2)^2 \ge 0$. To ∞!

Thus, limit is ∞.



2. [2 points each] Study the graph below of $y = f(x)$. Using the graph find the following limits.



a. $\lim_{x\to 0^-} f(x)$ does not exist

b. $\lim_{x\to 0+} f(x) = 2$

c. $\lim_{x\to 1} f(x) = 2$

d. $\lim_{x\to 2^-} f(x) = 2$

e. $\lim_{x\to 2+} f(x) = \infty$

3. [10 points] Find the equation of the line tangent to the graph of

$$y = 3x + 2\sqrt{x} = f(x)$$

at the point $(1, 5)$. Express your answer in slope-intercept form ($y = mx + b$).

$y - 5 = m(x-1)$

$m = f'(1)$. $f'(x) = 3 + \frac{1}{\sqrt{x}}$ (2's cancel)

$f'(1) = 3 + 1 = 4$

$y - 5 = 4(x-1)$

$y = 4x - 4 + 5$

$y = 4x + 1$

4. [10 points] Find the value of c for which the function below will be continuous for all real numbers.

$$f(x) = \begin{cases} \sin x & \text{for} \quad x \leq \pi \\ x + c & \text{for} \quad x > \pi \end{cases}$$

We need $\sin(\pi) = \pi + c$

$0 = \pi + c$

$c = -\pi$

5. [10 points] Show there is a solution to the equation $x = \cos x$ in the interval $(0, \pi/2)$. Hint: Let $f(x) = x - \cos x$ and apply the Intermediate Value Theorem to show $f(x) = 0$ at least once in the interval $(0, \pi/2)$.

$f(0) = 0 - \cos(0) = 0 - 1 = -1$

$f(\pi/2) = \frac{\pi}{2} - \cos(\frac{\pi}{2}) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$.

Since $-1 < 0 < \frac{\pi}{2}$, the IVT implies

there is a value $c \in (0, \frac{\pi}{2})$ such that $f(c) = 0$.

Then $c - \cos(c) = 0$ implies $c = \cos(c)$.

[c is called a fixed pt. of $\cos(x)$. Put your calculator into radian mode. Enter any number. Now press $\boxed{\cos}$ over and over. What happens?]

6. [10 points] Using the definition of the derivative and the rules for limits find the derivative of $f(x) = \sqrt{x+1}$.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}$$

$$= \lim_{h \to 0} \frac{(x+h+1) - (x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h+1} + \sqrt{x+1})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}}$$

$$= \frac{1}{\sqrt{x+0+1} + \sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}.$$

7. [10 points] You have a ball. You project the ball upward 20 ft/sec from a platform 30 ft above the ground. When will it hit the ground? (Assume the ball does not hit the platform on the way down, neglect air resistance and assume the acceleration due to gravity is 32 ft/sec/sec downward.)

$a = -32$

$V = -32t + V_0 = -32t + 20$

$S = -32\frac{t^2}{2} + 20t + S_0 = -16t^2 + 20t + 30.$

Solve $s(t) = 0$.

$$t = \frac{-20 \pm \sqrt{400 - 4(-16)30}}{-32} = \frac{20 \mp \sqrt{2320}}{32}$$

$= -0.88$ seconds ← disgard

$\boxed{2.13 \text{ seconds}}$

8. [5 points each] Find the derivatives.

a. $f(x) = x^2 \sin 3x$

$f'(x) = (x^2)' \sin(3x) + x^2(\sin(3x))'$

$= 2x \sin(3x) + x^2 \cos(3x) \cdot \underbrace{(3x)'}_{=3}$

$= 2x \sin(3x) + 3x^2 \cos(3x)$

b. $b(x) = \frac{x+1}{\cos^2 x}$

$b' = \dfrac{(x+1)' \cos^2 x - (x+1)(\cos^2 x)'}{\cos^4 x}$

$= \dfrac{\cos^2 x - (x+1)\, 2\cos x (\cos x)'}{\cos^4 x}$

$= \dfrac{\cos^2 x + 2(x+1)\cos x \sin x}{\cos^4 x}$

$= \dfrac{\cos x + 2(x+1)\sin x}{\cos^3 x}$

Leaving it like this is fine

c. $g(x) = \sqrt{x}(\tan x)$

$g'(x) = (\sqrt{x})' \tan x + \sqrt{x}(\tan x)'$

$= \dfrac{\tan x}{2\sqrt{x}} + \sqrt{x} \sec^2 x$

d. $p(x) = 4x^3 + \sec(x^2)$

$p'(x) = 12x^2 + \sec(x^2)\tan(x^2)\underbrace{(x^2)'}_{=2x}$

$= 12x^2 + 2x \sec(x^2)\tan(x^2)$

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1. [5 points each] Find the following limits. Show your work.

a. $\lim_{x\to 0} \frac{\tan 3x}{x}$ $= \lim_{x\to 0} \frac{\sin 3x}{x} \cdot \frac{1}{\cos 3x}$

$= 3 \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \lim_{x\to 0} \frac{1}{\cos x}$

$= 3 \cdot 1 \cdot 1 = 3$

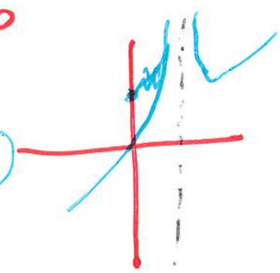
b. $\lim_{x\to 2} \frac{x^3}{x^2-4x+4}$ (Hint: Factor the denominator.)

$= \lim_{x\to 2} \frac{x^3}{(x-2)^2} = \frac{\text{"}8\text{"}}{0^2}$

near $x=2$ $x^3 \approx 8 > 0$

and $(x-2)^2 > 0$.

Thus limit is ∞



c. $\lim_{x\to\infty} 2x - \sqrt{4x^2+x}$ $\frac{2x+\sqrt{4x^2+x}}{2x+\sqrt{4x^2+x}}$

$= \lim_{x\to\infty} \frac{4x^2-(4x^2+x)}{2x+\sqrt{4x^2+x}} = \lim_{x\to\infty} \frac{-x}{2x+\sqrt{4x^2+x}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$

$= \lim_{x\to\infty} \frac{-1}{2+\sqrt{4+\frac{1}{x}}} = \frac{-1}{2+\sqrt{4+0}} = -\frac{1}{4}$

d. $\lim_{t\to\infty} \frac{\cos 2t}{t}$ $= 0$

$-1 \le \cos 2t \le 1$

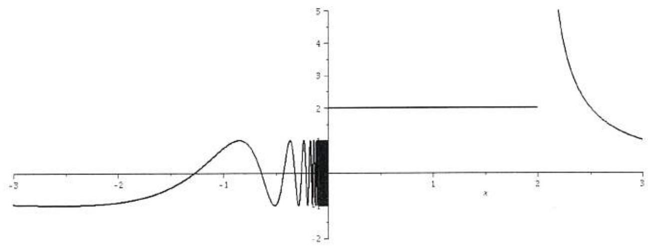
$-\frac{1}{t} \le \frac{\cos 2t}{t} \le \frac{1}{t}$

$-\frac{1}{t}$ and $\frac{1}{t} \to 0$.

By Squeeze Thm

$\lim_{t\to\infty} \frac{\cos 2t}{t} = 0$.

2. [2 points each] Study the graph below of $y = f(x)$. Using the graph find the following limits.



a. $\lim_{x \to 2^-} f(x)$ $= 2$

b. $\lim_{x \to 2^+} f(x)$ $= \infty$

c. $\lim_{x \to 1} f(x)$ 2

d. $\lim_{x \to 0^-} f(x)$ does not exist

e. $\lim_{x \to 0^+} f(x)$ 2

3. [10 points] Find the equation of the line tangent to the graph of

$$y = 3x + 2\sqrt{x} = f(x)$$

at the point $(1, 5)$. Express your answer in slope-intercept form ($y = mx + b$).

$y - 5 = m(x - 1)$

$m = f'(1) =$ $f'(x) = 3 + \frac{1}{\sqrt{x}}$ (The 2's cancel.)

$f'(1) = 3 + \frac{1}{\sqrt{1}} = 4.$

$y - 5 = 4(x - 1)$

$y = 4x - 4 + 5$

$y = 4x + 1$

4. [10 points] Show there is a solution to the equation $x = \cos x$ in the interval $(0, \pi/2)$. Hint: Let $f(x) = x - \cos x$ and apply the Intermediate Value Theorem to show $f(x) = 0$ at least once in the interval $(0, \pi/2)$.

$$f(0) = 0 - \cos(0) = -1.$$

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$-1 < 0 < \frac{\pi}{2}$$

By the IVT, there exists a c in $\left(0, \frac{\pi}{2}\right)$ such that $f(c) = 0$.

But $c - \cos(c) = 0$, means $\cos(c) = c$.

[Put your calculator into radian mod.
Enter any number. Press $\boxed{\cos}$ over and over. what happens?]

5. [10 points] Find the value of c for which the function below will be continuous for all real numbers.

$$f(x) = \begin{cases} \sin x & \text{for} \quad x \leq \pi \\ x - c & \text{for} \quad x > \pi \end{cases}$$

Need $\sin(\pi) = \pi - c$.

Thus, $0 = \pi - c$

Hence $c = \pi$.

6. [10 points] Using the definition of the derivative and the rules for limits find the derivative of $f(x) = \sqrt{x-1}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h-1} - \sqrt{x-1}}{h} \cdot \frac{\sqrt{x+h-1} + \sqrt{x-1}}{\sqrt{x+h-1} + \sqrt{x-1}}$$

$$= \lim_{h \to 0} \frac{(\cancel{x} + h - \cancel{1}) - (\cancel{x} - \cancel{1})}{h\left(\sqrt{x+h-1} + \sqrt{x-1}\right)} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{} + \sqrt{}\right)}$$

$$= \frac{1}{\sqrt{x+0-1} + \sqrt{x-1}} = \frac{1}{2\sqrt{x-1}}$$

7. [10 points] You have a ball. You project the ball upward 20 ft/sec from a platform 30 ft above the ground. When will it hit the ground? (Assume the ball does not hit the platform on the way down, neglect air resistance and assume the acceleration due to gravity is 32 ft/sec/sec downward.)

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$s = \frac{-32}{2} t^2 + 20t + \underbrace{30}_{s_0}$

$= -16t^2 + 20t + 30$

Solve, $S(t) = 0$.

$$t = \frac{-20 \pm \sqrt{400 - 4(-16)30}}{-32}$$

$$= \frac{20 \mp \sqrt{2320}}{32} = \begin{array}{l} -0.88 \text{ seconds} \Leftarrow \text{disgard.} \\ 2.13 \text{ seconds.} \end{array}$$

8. [5 points each] Find the derivatives.

a. $f(x) = x^2 \sin 5x$

$f' = (x^2)' \sin(5x) + x^2 (\sin(5x))'$

$= 2x \sin(5x) + x^2 \cos(5x) \cdot \underbrace{(5x)'}_{=5}$

$= 2x \sin(5x) + 5x^2 \cos(5x)$

b. $b(x) = \frac{x+1}{\cos^2 x}$

$b' = \dfrac{\overbrace{(x+1)'}^{=1} \cos^2 x - (x+1)(\cos^2 x)'}{\cos^4 x}$

$(\cos^2 x)' = 2\cos x \cdot (\cos x)'$

$= -2 \cos x \sin x.$

Thus, $b'(x) = \dfrac{\cos^2 x + 2(x+1)\cos x \sin x}{\cos^4 x}$

$= \dfrac{\cos x + 2(x+1)\sin x}{\cos^3 x}$

or

Leaving it like this is fine.

c. $g(x) = \sqrt{x}(\tan x)$

$g'(x) = (\sqrt{x})' \tan x + \sqrt{x} (\tan x)'$

$= \dfrac{\tan x}{2\sqrt{x}} + \sqrt{x} \sec^2 x$

d. $h(x) = \sec(x^2) + 3x^4$

$h' = \sec(x^2) \tan(x^2) \cdot [x^2]' + 12x^3$

$= 2x \sec(x^2) \tan(x^2) + 12x^3.$