Name: ______________________________

Math 150 Test 2 Fall 2024

ONLY SCIENTIFIC CALCULATORS ALLOWED

(1) [5 points each] Find the following limits. Show your work.

a. $\lim_{x\to 0^+} x\ln x$ Has form $0\cdot(-\infty)$

$\frac{\infty}{\infty}$

$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = -\lim_{x\to 0^+} x = 0$

b. $\lim_{x\to 0} \frac{\sinh 3x}{\tan 2x}$ $\frac{0}{0}$ $= \lim_{x\to 0} \frac{3\cosh 3x}{2\sec^2 2x} = \frac{3\cdot 1}{2\cdot 1^2} = \frac{3}{2}$

Form 0^0

c. $\lim_{x\to 0^+} x^x = L$

$\ln L = \ln \lim_{x\to 0^+} x^x = \lim_{x\to 0^+} \ln x^x$

$= \lim_{x\to 0^+} x\ln x = 0$

by (a).

Thus $\ln L = 0$.

Thus $L = e^0 = 1$

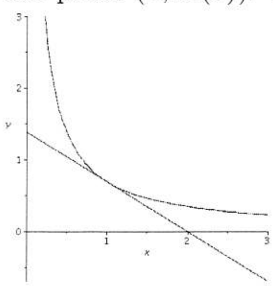
d. $\lim_{x\to 7} \frac{x^2-49}{x-7}$ $\frac{0}{0}$ $= \lim_{x\to 7} x+7 = 14$

or $= \lim_{x\to 7} \frac{2x}{1} = 14$.

(2) [10 points] Find the equation of the line tangent to the graph of

$$e^{xy} = 2$$

at the point $(1, \ln(2))$. Express your answer in slope-intercept form ($y = mx + b$).



Method I Apply $\frac{d}{dx}$ to both sides to get

$$e^{xy}(y + xy') = 0$$

$$y + xy' = 0$$

$$y' = -\frac{y}{x} = -\ln 2$$

$$y - \ln 2 = -\ln 2\,(x - 1)$$

$$y = -\ln 2\, x + 2\ln 2$$

Method II. Solve for y.

$$xy = \ln 2$$

$$y = \frac{\ln 2}{x}$$

$$y' = \frac{-\ln 2}{x^2} = -\ln 2.$$

Rest is the same.

(3) [10 points] Derive the formula for the derivative of $\arctan x$. Hint: Let $\theta = \arctan x$.

$$\tan\theta = x$$

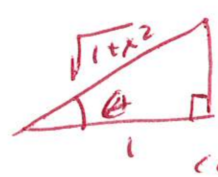
Apply $\frac{d}{dx}$.

$$\sec^2\theta \cdot \theta' = 1$$

$$\theta' = \cos^2\theta$$

$$\theta' = \frac{1}{1+x^2}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$



$$\cos\theta = \frac{1}{\sqrt{1+x^2}}$$

(4) [5 points] 10% of a radioactive substance decays after 50 minutes. What is its half-life? Give the exact answer, and the numerical approximation to 3 decimal places.

$e^{rt} = .9$ when $t = 50$

$r50 = \ln .9$

$r = \frac{\ln .9}{50} \approx -0.00210721$

$e^{rt} = .5$

$t_{\frac{1}{2}} = \frac{\ln(.5)}{r} \approx 328.941$ min

(5) [5 points] A population of bacteria doubles every 4 hours. How long does in take for this population to triple? Give the exact answer, and the numerical approximation to 3 decimal places.

$e^{rt} = 2$ when $t = 4$.

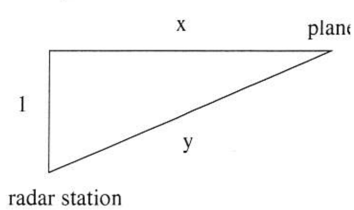
$r4 = \ln 2$

$r = \frac{\ln 2}{4}$

$e^{rt} = 3$ $\quad rt = \ln 3$

$t = \ln 3 / r = 4\frac{\ln 3}{\ln 2} \approx 6.340$ hours

(6) [10 points] A plane is flying horizontally at a fixed altitude of 1 mile. Its speed is a constant 500 miles/hour. It passes directly over a radar station. Find the rate at which the distance between the plane and the radar station is increasing when the plane is 2 miles from the radar station. See figure below.



$x' = 500$ $\quad$ Find y' when $y = 2$.

$x^2 + 1^2 = y^2$

$\cancel{2}xx' = \cancel{2}yy'$

$y' = \frac{x x'}{y} = \frac{500x}{2}$ $\quad$ need x when $y = 2$.

$x^2 = y^2 - 1 = 3$

$x = \sqrt{3}$

$= 250\sqrt{3}$

≈ 433 mph

This is #13 from Section 2.7

(7) [12 points] Let $f(x) = x^2 - x - \ln x$, for $x > 0$. Our goal is to sketch a rough graph of $y = f(x)$.

a. [1 point] What is $f(1)$? b. [2 points] What is $\lim_{x\to 0+} f(x)$? ∞ What is $\lim_{x\to\infty} f(x)$? ∞

[Brief justifications are sufficient.]

$f(1) = 1 - 1 - \ln 1 = 0 - 0 = 0$

$x^2 - x \to 0$, but $-\ln x \to \infty$.

x^2 grows faster than x or $\ln x$.

c. [3 points] Find $f'(x)$. For which value(s) of x in the domain is $f'(x) = 0$.

$f'(x) = 2x - 1 - \frac{1}{x} = 0 \Rightarrow 2x^2 - x - 1 = 0$

$(2x+1)(x-1) = 0$ $x = -\frac{1}{2}, 1$

only $x = 1$ is in domain.

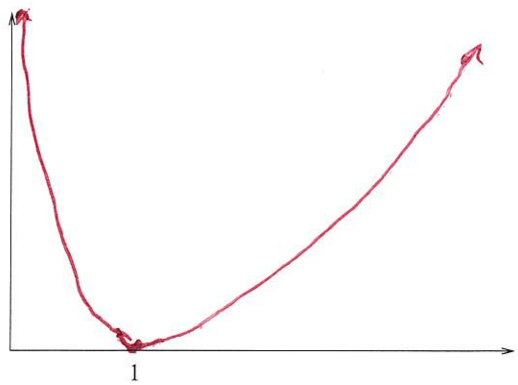
d. [2 points] On which open interval(s) will the graph of $y = f(x)$ be increasing? Decreasing?

$\searrow (0,1)$ $\nearrow (1, \infty)$

e. [2 points] Find $f''(x)$. Does the graph of $y = f(x)$ have any inflection points?

$f''(x) = 2 + \frac{1}{x^2}$. Never 0. No infl. pts.

f. [2 points] Sketch a rough graph of $y = f(x)$.



(8) [4 points each] The velocity of an object is given by

$$v(t) = 3e^{2t} - \cosh(3x) + \frac{2}{t+1}$$

a. Find the acceleration $a(t)$. b. Find the position, $s(t)$, assuming $s(0) = 3$.

$a = 6e^{2t} - 3\sinh(3x) - \frac{2}{(t+1)^2}$

$s = \frac{3}{2}e^{2t} - \frac{1}{3}\sinh(3x) + 2\ln(t+1) + C$

$s(0) = \frac{3}{2} - 0 + 2\ln(1) + C$, $\ln(1) = 0$

$C = 3 - \frac{3}{2} = \frac{3}{2}$.

(9) [5 points each] Find the derivatives.

a. $(\sinh(\sin(x^3)))'$

$\cosh(\sin(x^3)) \cdot \cos(x^3) \cdot 3x^2$

b. $(\cos(\ln(3x^5)))'$

$-\sin(\ln(3x^5)) \frac{1}{3x^5} \cdot 15x^4$

$= -\frac{5}{x} \sin(\ln(3x^5))$

c. $(\tan^3(x^2))'$

$3\tan^2(x^2) \cdot \sec^2(x^2) \cdot 2x$

$= 6x \tan^2(x^2) \sec^2(x^2)$

d. $(x^{\cos x})'$

let $y = x^{\cos x}$.

$\ln y = \ln x^{\cos x} = \cos x \ln x$

$\frac{y'}{y} = -\sin x \ln x + \frac{\cos x}{x}$

$y' = (x^{\cos x})\left[\frac{\cos x}{x} - \sin x \ln x\right]$