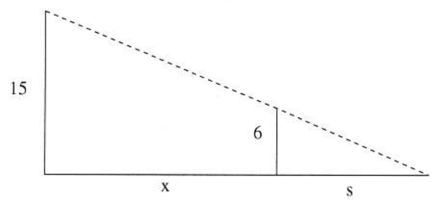
Math 150 Worksheet 9 Fall 2024

Name: ______________________

1. [10 points] A street light is on a post 15 ft above a sidewalk. A 6 ft tall person walks away from the light post at 5 ft/sec. How fast is the person's shadow growing when the person is 20 from the light post?



Use similar triangles.

$$\frac{6}{15} = \frac{s}{x+s}$$

$$6x + 6s = 15s$$

$$6x = 9s$$

$$s = \frac{2}{3}x$$

Thus $\frac{ds}{dt} = \frac{2}{3}\frac{dx}{dt} = \frac{2}{3}\cdot 5 = \frac{10}{3}$ ft/sec.

Notice: we did not need to use $x=20$!

2. [5 points] If a population increases by 30% in 60 minutes, when will it double?

$$P(t) = P_0 e^{rt}$$

$$e^{r(60)} = 1.3$$

$$r = \frac{\ln(1.3)}{60}$$

$$e^{rt} = 2$$

$$t = \frac{\ln 2}{r} = \frac{60 \ln 2}{\ln(1.3)}$$

≈ 158.5 minutes.

3. [5 points] If the half-life of a radioactive substance is 3700 years, when will it decrease to 10% of its original amount?

$$e^{r\,3700} = \frac{1}{2}$$

$$r = \frac{-\ln(2)}{3700}$$

$$e^{rt} = .1$$

$$t = \frac{\ln(.1)}{r} = \frac{3700 \ln 10}{\ln 2}$$

$\approx 12{,}291$ years

4. [10 points] Let $f(x) = x^5 + x^3 + x$. Use the Intermediate Value Theorem (IVT) to show $f(x) = 1$ at least once. Use the Mean Value Theorem (MVT) to show $f(x)$ cannot equal 1 more than once.

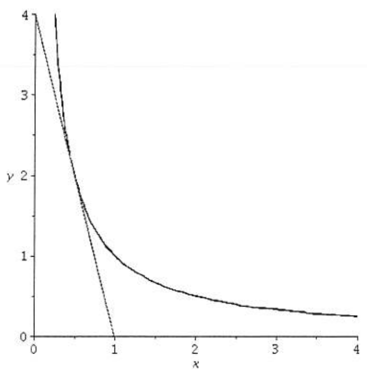
$f(0) = 0$, $f(1) = 3$. f is continuous on $[0,1]$.
Since $f(0) < 1 < f(1)$, by the IVT there exists at least one value $c \in (0,1)$ such that $f(c) = 1$.

Suppose a and b are two distinct values such that $f(a) = f(b) = 1$. By the MVT there exists a number c in between a and b where

$$f'(c) = \frac{f(b) - f(a)}{b-a} = 0.$$ (using that $f(x)$ is diff. on all of $\mathbb{R}$.)

But $f'(x) = 5x^4 + 3x^2 + 1$ is always > 0. This is a contradiction. Thus, there cannot be 2 distinct values where $f(x) = 1$.

5. [10 points] Below is the graph of the relation $\sinh xy = C$, where $C = (e - 1/e)/2 \approx 1.175201194$. Find the equation of the line tangent to the graph at the point $(1/2, 2)$.



Apply $\frac{d}{dx}$ to both sides of $\sinh xy = C$.
Thus,

$$\cosh(xy)(y + xy') = 0.$$

Since $\cosh(xy)$ is never 0, we have

$$y + xy' = 0$$

so $y' = \frac{-y}{x} = \frac{-2}{\frac{1}{2}} = -4$

The line is

$$y - 2 = -4\left(x - \tfrac{1}{2}\right)$$

or $y = -4x + 4$

6. [11 points] Let $f(x) = \frac{4x}{x^2+4}$. Our goal is to sketch a rough graph of $y = f(x)$.

a. When is $f(x) = 0$? Is $f(x)$, even, odd or either? (2 points)

b. What are the limits of $f(x)$ as $x \to \pm\infty$? (2 points)

c. Find $f'(x)$. When is it 0? On which open intervals is $f(x)$ increasing? Decreasing? (3 points)

d. What are the maximum and minimum values of $f(x)$? (2 points)

e. Sketch a graph of $y = f(x)$. (5 points)

a. $f(0)=0$. f is odd.

b. $\lim_{x\to\infty} \frac{4x}{x^2+4} = \lim_{x\to\infty} \frac{\frac{4}{x}}{1+\frac{4}{x^2}} = \frac{0}{1+0} = 0$. Likewise for $x \to -\infty$.

c. $f'(x) = \frac{4(x^2+4) - 4x(2x)}{(x^2+4)^2} = \frac{-4x^2+16}{(x^2+4)^2}$.

$f'(x) = 0 \iff x = \pm 2$.

f is dec. on $(-\infty,-2)$ and $(2,\infty)$

f is inc. on $(-2,2)$.

d. $f(2) = \frac{8}{8} = 1$

$f(-2) = \frac{-8}{8} = -1$.

e.