Example. Let $f(x) = \frac{e^{2x}+2}{e^x+1}$. Graph and describe behavior for large values of $|x|$.

Solution. This function is never zero and has no singularities. We compute the limits as $x \to \pm\infty$.

$$\lim_{x\to-\infty} \frac{e^{2x}+2}{e^x+1} = \frac{0+2}{0+1} = 2.$$

Thus, $y = 2$ is the backward horizontal asymptote.

$$\lim_{x\to\infty} \frac{e^{2x}+2}{e^x+1} = \lim_{x\to\infty} \frac{e^x+2e^{-x}}{1+e^{-x}} = \infty.$$

But, we can say more. For $x \gg 0$ it seems that $f(x) \approx e^x$. Let's check this.

$$\begin{aligned}
\lim_{x\to\infty} f(x) - e^x &= \lim_{x\to\infty} \frac{e^{2x}+2}{e^x+1} - \frac{e^x(e^x+1)}{e^x+1} \\
&= \lim_{x\to\infty} \frac{2-e^x}{e^x+1} \\
&= \lim_{x\to\infty} \frac{2e^{-x}-1}{1+e^{-x}} \\
&= \frac{0-1}{1+0} = -1.
\end{aligned}$$

So, our hunch was not quite right. But, it is true that

$$\lim_{x\to\infty} f(x) - (e^x - 1) = 0.$$

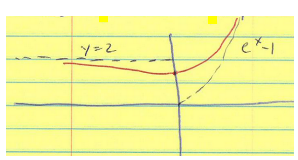
Thus, we can approximate $f(x)$ by $e^x - 1$ for large positive values of x.

Here is another way to see this.

$$\begin{array}{r l}
 & \quad e^x - 1 \\
e^x + 1 & \big) \; e^{2x} + 2 \\
 & \underline{-(e^{2x} + e^x)} \\
 & \quad -e^x + 2 \\
 & \quad \underline{-(-e^x - 1)} \\
 & \qquad\quad 3
\end{array}$$

Thus, $f(x) = e^x - 1 + \frac{3}{e^x+1}$. For $x \gg 0$ we know $\frac{3}{e^x+1} \approx 0$, so $f(x) \approx e^x - 1$.

Now, $f(0) = 3/2$, so a rough sketch might look like this.



But, maybe the graph crosses $y = 2$ for some value $x < 0$, like this.

We will show that this does not happen. Suppose $f(x) = 2$. Then

$$\begin{aligned}
\frac{e^{2x}+2}{e^x+1} &= 2 \\
e^{2x}+2 &= 2e^x+2 \\
e^{2x} &= 2e^x \\
e^x &= 2 \\
x &= \ln 2 \approx 0.693 > 0.
\end{aligned}$$

Thus, the graph $y = f(x)$ does not cross the line $y = 2$ for $x < 0$.

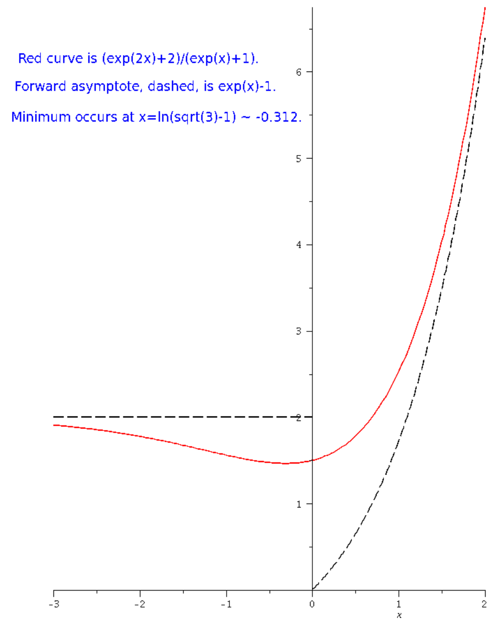
Could $y = f(x)$ cross the graph of $y = e^x - 1$ for some $x > 0$? Let's check.

$$\begin{aligned}
\frac{e^{2x}+2}{e^x+1} &= e^x - 1 \\
e^{2x}+2 &= e^{2x}-1 \\
2 &= -1.
\end{aligned}$$

This, of course, never happens.

It is not clear where the minimum of $y = f(x)$ occurs. It our rough sketch it looks like the minimum occurs at $x = 0$, but we cannot be certain of this. Later, using the derivative, we will be able to show the minimum is at $x = \ln(\sqrt{3} - 1) \approx -0.312$.

Below is a computer graph of $y = f(x)$.



□