Example. Find $\lim_{x\to\infty} \sqrt{x^2+x+1} - x$.

Solution.

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+x+1} - x &= \lim_{x\to\infty} \frac{\sqrt{x^2+x+1} - x}{1} \cdot \frac{\sqrt{x^2+x+1} + x}{\sqrt{x^2+x+1} + x} \\
&= \lim_{x\to\infty} \frac{(x^2+x+1) - x^2}{\sqrt{x^2+x+1} + x} \\
&= \lim_{x\to\infty} \frac{x+1}{\sqrt{x^2+x+1} + x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} + 1} \\
&= \frac{1+0}{\sqrt{1+0+0}+1} = \frac{1}{1+1} = \frac{1}{2}.
\end{aligned}$$

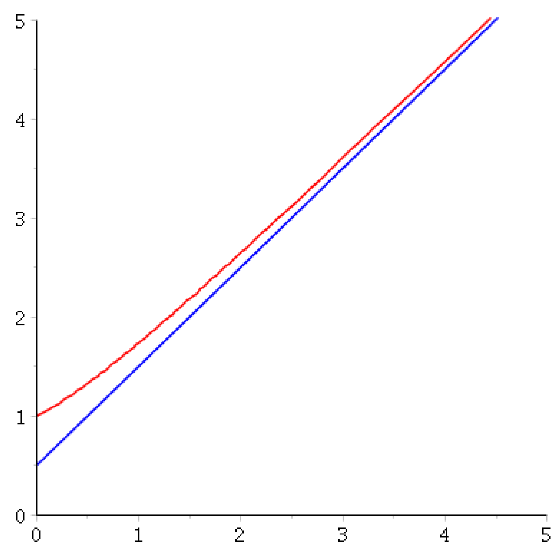
□

Question. Why is this interesting?

Answer. If follows that

$$\lim_{x\to\infty} \sqrt{x^2+x+1} - (x + \tfrac{1}{2}) = 0.$$

Thus, for $x \gg 0$ we have $\sqrt{x^2+x+1} \approx x + \frac{1}{2}$. That is, we have found an *oblique asymptote*. Therefore, if you had an application involving $\sqrt{x^2+x+1}$, and the application only involved large positive values of x, you could just use $x + \frac{1}{2}$ instead, making your life much easier. Below is a graph comparing these two functions.



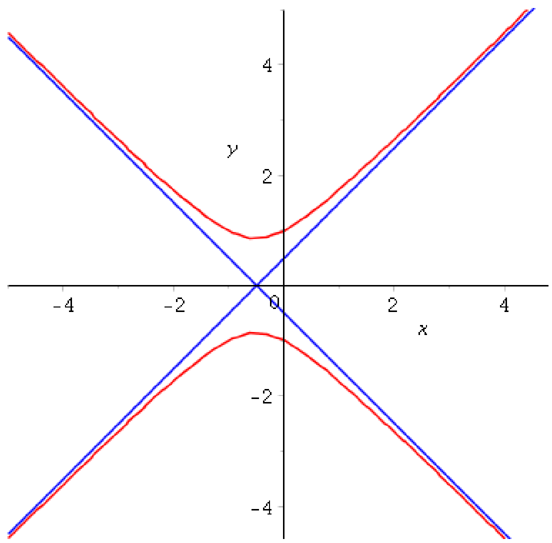
What else? Well, $y = \sqrt{x^2 + x + 1}$ is our old friend the hyperbola!

$$\begin{aligned} y^2 &= x^2 + x + 1 \\ y^2 &= x^2 + x + \tfrac{1}{4} + \tfrac{3}{4} \\ y^2 &= (x + \tfrac{1}{2})^2 + \tfrac{3}{4}. \end{aligned}$$

Or,

$$y^2 - (x + \tfrac{1}{2})^2 = \tfrac{3}{4}.$$

See below for a graph of this hyperbola with its oblique asymptotes.



Now, do an internet search for: 'Oumuamua asteroid trajectory.