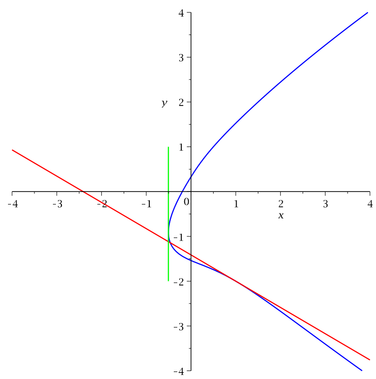
Implicit Derivative Example. Consider the relation

$$y^4 + 3y - 4x^3 = 5x + 1.$$

Its graph is shown below, along with two tangent lines to be discussed.



A. Find the tangent line to the graph at the point $(1, -2)$. See the red line in the graph above.

B. Find the coordinates of the point where the tangent line would be vertical. See the green line in the graph above.

Solution for part A. It is easy to verify that $x = 1$, $y = -2$ does satisfy the given relation. Do this. It would be very difficult to solve explicitly for y. From the graph however it seems clear that we can think of $y(x)$ as an implicit function of x near the point $(1, -2)$. Apply d/dx to both sides of the relation to get

$$4y^3y' + 3y' - 12x^2 = 5.$$

Plug in $(1, -2)$ to get

$$-32y' + 3y' - 12 = 5.$$

Thus, $y' = -\frac{17}{29}$. This is the slope of our tangent line. Thus, the tangent line is given by

$$y + 2 = -\tfrac{17}{29}(x - 1)$$

or

$$y = -\tfrac{17}{29}x - \tfrac{41}{29}.$$

It is the red line in the graph shown above.

□

Solution to part B. Now we need to think of x as a function of y, and then to find where $\frac{dx}{dy} = 0$. Apply d/dy to both sides of the original relation to get

$$4y^3 + 3 - 12x^2x' = 5x'$$

where x' means $\frac{dx}{dy}$.

Set $x' = 0$. This gives $4y^3 + 3 = 0$, so

$$y = -\sqrt[3]{\frac{3}{4}} \approx -0.90856.$$

Next we need to find x. We can rewrite the original relation as

$$4x^3 + 5x + 1 - [y^4 + 3y] = 0.$$

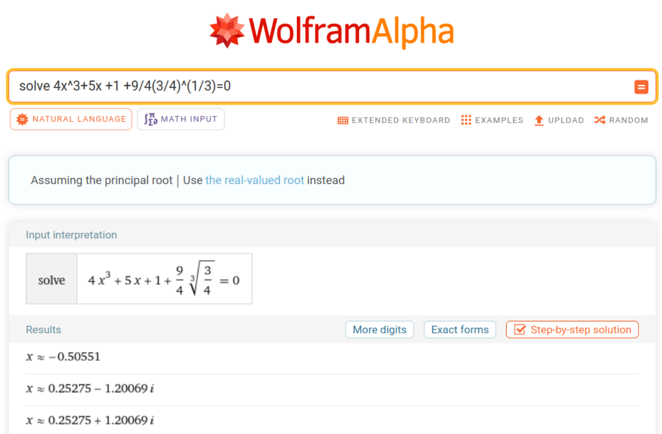
Substituting in for y gives

$$4x^3 + 5x + 1 - \left[\left(-\sqrt[3]{\frac{3}{4}}\right)^4 - 3\sqrt[3]{\frac{3}{4}}\right] = 0$$

or

$$4x^3 + 5x + 1 + \tfrac{9}{4}\sqrt[3]{\frac{3}{4}} = 0.$$

Solving a cubic is messy, so we will use an online solver. See below.



`https://www.wolframalpha.com/`

It gives $x \approx -0.50551$. (We can ignore the complex roots.) Thus, in the graph above the point of tangency between the vertical green line and our curve is about $(-0.50551, -0.90856)$. $\square$

Notes: Part A of this problem appeared in *Calculus With Analytic Geometry*, by Earl Swokowski, 1975. The Graph was made with Maple.