Limit Examples
(Pre-L'Hôpital's Rule)

Simply reading these will not help you. You should work them out for yourself, filling in any gaps, looking up facts about functions as needed, and so on. That is, you need to think your way through them.

(1) $\lim_{\theta\to 0} \theta \csc\theta = \lim_{\theta\to 0} \frac{\theta}{\sin\theta} = \frac{1}{\lim_{\theta\to 0} \frac{\sin\theta}{\theta}} = \frac{1}{1} = 1.$

(2) $\lim_{\theta\to 0} \frac{\tan 2\theta \sin^2\theta}{\theta^3} = \lim_{\theta\to 0} \frac{2}{2} \frac{\sin 2\theta}{\theta\cos 2\theta} \frac{\sin\theta}{\theta} \frac{\sin\theta}{\theta} = 2\lim_{\theta\to 0} \frac{\sin 2\theta}{2\theta} \frac{1}{\cos 2\theta} \frac{\sin\theta}{\theta} \frac{\sin\theta}{\theta} = 2\cdot 1\cdot \frac{1}{1} \cdot 1 \cdot 1 = 2.$

(3) $\lim_{\theta\to 0} \frac{\cos 3\theta}{\theta} =$ "$\frac{1}{0}$" is undefined. (The left and right limits are $-\infty$ and ∞, respectively.)

(4) $\lim_{\theta\to\pi/2} \frac{\cos\theta}{\theta - \pi/2} =?$ Recall $\cos\theta = -\sin\left(\theta - \frac{\pi}{2}\right)$. Let $q = \theta - \frac{\pi}{2}$.

Thus, $? = \lim_{\theta\to\pi/2} \frac{-\sin\left(\theta - \frac{\pi}{2}\right)}{\theta - \frac{\pi}{2}} = \lim_{q\to 0} \frac{-\sin q}{q} = -1.$

(5) $\lim_{\theta\to 0} \frac{\theta\sec\theta}{\csc\theta} = \lim_{\theta\to 0} \frac{\theta\sin\theta}{\cos\theta} = \frac{0\cdot 0}{1} = 0.$

(6) $\lim_{\theta\to 0} \frac{\sin 3\theta}{\sin\theta} = \lim_{\theta\to 0} \frac{3}{3}\frac{\theta}{\theta}\frac{\sin 3\theta}{\sin\theta} = 3\lim_{\theta\to 0} \frac{\sin 3\theta}{3\theta}\frac{\theta}{\sin\theta} = 3\cdot 1\cdot 1 = 3.$

(7) $\lim_{\theta\to 0} \frac{\tan\theta}{\sin 2\theta} = \lim_{\theta\to 0} \frac{\tan\theta}{2\sin\theta\cos\theta} = \frac{1}{2}\lim_{\theta\to 0} \frac{1}{\cos^2\theta} = \frac{1}{2}\cdot\frac{1}{1^2} = \frac{1}{2}.$

(8) $\lim_{\theta\to 0} \frac{\theta}{\theta + \tan\theta} = \lim_{\theta\to 0} \frac{\theta}{\theta + \tan\theta}\frac{\frac{1}{\theta}}{\frac{1}{\theta}} = \lim_{\theta\to 0} \frac{1}{1 + \frac{\tan\theta}{\theta}} = \frac{1}{1+1} = \frac{1}{2}.$

(9) $\lim_{x\to\infty} \cos\arctan x = \cos\frac{\pi}{2} = 0.$

(10) $\lim_{x\to\infty} \frac{\tan x}{x}$ is undefined. Why?

(11) $\lim_{x\to\infty} \frac{\frac{1}{x} + \sin\theta}{\cos\theta} = \frac{0 + \sin\theta}{\cos\theta} = \tan\theta.$ (Since θ does not depend on x.)

(12) $\lim_{x\to\pi^+} \frac{x\sin x + \csc x}{3x} = \lim_{x\to\pi^+} \frac{\sin x}{3} + \frac{\csc x}{3x} = \frac{0}{3} + \frac{-\infty}{3\pi} = -\infty.$

(13) $\lim\limits_{y\to-\pi/2^-} \dfrac{\cot y - \tan y}{y} = \dfrac{0-\infty}{-\pi/2} = \infty.$

(14) $\lim\limits_{x\to\infty} \dfrac{x+4}{x^2-16} = 0.$

(15) $\lim\limits_{x\to 4} \dfrac{x+4}{x^2-16} = \lim\limits_{x\to 4} \dfrac{1}{x-4}$ is undefined. (The left and right limits are $-\infty$ and ∞, respectively.)

(16) $\lim\limits_{x\to -4} \dfrac{x+4}{x^2-16} = \lim\limits_{x\to -4} \dfrac{1}{x-4} = -\dfrac{1}{8}.$

(17) $\lim\limits_{x\to 0} \dfrac{x+4}{x^2-16} = \dfrac{0+4}{0^2-16} = -\dfrac{1}{4}.$

(18) $\lim\limits_{x\to\infty} \dfrac{4e^x+8}{3e^x+7} = \lim\limits_{x\to\infty} \dfrac{4e^x+8}{3e^x+7}\dfrac{e^{-x}}{e^{-x}} = \lim\limits_{x\to\infty} \dfrac{4+8e^{-x}}{3+7e^{-x}} = \dfrac{4+0}{3+0} = \dfrac{4}{3}.$

(19) $\lim\limits_{x\to -\infty} \dfrac{4e^x+8}{3e^x+7} = \dfrac{0+8}{0+7} = \dfrac{8}{7}.$

(20) $\lim\limits_{x\to\infty} \arctan \dfrac{x^2+x+1}{x^2+3x+4} = \arctan \lim\limits_{x\to\infty} \dfrac{x^2+x+1}{x^2+3x+4} \arctan 1 = \dfrac{\pi}{4}.$

(21) $\lim\limits_{x\to\infty} \dfrac{\sqrt{6x^3+x}}{x^2+2} = \lim\limits_{x\to\infty} \sqrt{\left(\dfrac{\sqrt{6x^3+x}}{x^2+2}\right)^2} = \sqrt{\lim\limits_{x\to\infty} \dfrac{6x^3+x}{x^4+4x+4}} = \sqrt{0} = 0.$

(22) $\lim\limits_{x\to\infty} \dfrac{x^2+x+7}{\sqrt{x^4+8}} = \sqrt{\lim\limits_{x\to\infty} \dfrac{x^4+2x^3+15x^2+14x+49}{x^4+8}} = \sqrt{1} = 1.$

Or, $\lim\limits_{x\to\infty} \dfrac{x^2+x+7}{\sqrt{x^4+8}} = \lim\limits_{x\to\infty} \dfrac{x^2+x+7}{\sqrt{x^4+8}}\dfrac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim\limits_{x\to\infty} \dfrac{1+\frac{1}{x}+\frac{7}{x^2}}{\sqrt{1+\frac{8}{x^4}}} = \dfrac{1}{\sqrt{1}} = 1.$

(23) $\lim\limits_{x\to\infty} \sqrt{17x^2+7} - 4x = \lim\limits_{x\to\infty} \dfrac{\sqrt{17x^2+7}-4x}{1}\dfrac{\sqrt{17x^2+7}+4x}{\sqrt{17x^2+7}+4x} =$
$\lim\limits_{x\to\infty} \dfrac{17x^2+7-16x^2}{\sqrt{17x^2+7}+4x} = \lim\limits_{x\to\infty} \dfrac{x^2+7}{\sqrt{17x^2+7}+4x}\dfrac{\frac{1}{x}}{\frac{1}{x}} = \lim\limits_{x\to\infty} \dfrac{x+\frac{7}{x}}{\sqrt{17+\frac{7}{x}}+4} =$
$\lim\limits_{x\to\infty} \dfrac{\infty+0}{\sqrt{17+0}+4} = \infty.$

(24) $\lim_{x\to\infty} \sqrt{16x^2+7} - 4x = \lim_{x\to\infty} \frac{\sqrt{16x^2+7}-4x}{1} \frac{\sqrt{16x^2+7}+4x}{\sqrt{16x^2+7}+4x} =$

$\lim_{x\to\infty} \frac{16x^2+7-16x^2}{\sqrt{16x^2+7}+4x} = \lim_{x\to\infty} \frac{7}{\sqrt{16x^2+7}+4x} \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{7}{x}}{\sqrt{16+\frac{7}{x}}+4} =$

$\lim_{x\to\infty} \frac{0}{\sqrt{16+0}+4} = 0.$

(25) $\lim_{x\to\infty} \sqrt{16x^2+x} - 4x = \lim_{x\to\infty} \frac{\sqrt{16x^2+x}-4x}{1} \frac{\sqrt{16x^2+x}+4x}{\sqrt{16x^2+x}+4x} =$

$\lim_{x\to\infty} \frac{16x^2+x-16x^2}{\sqrt{16x^2+x}+4x} = \lim_{x\to\infty} \frac{x}{\sqrt{16x^2+x}+4x} \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{16+\frac{1}{x}}+4} =$

$\lim_{x\to\infty} \frac{1}{\sqrt{16+0}+4} = \frac{1}{8}.$

An Application. Graph $y = \frac{x\sqrt{2+\frac{3}{x^2}}}{7x+4}$.

Solution. First we observe that there is a vertical asymptote at $x = -4/7$. Also notice that the function is undefined at $x = 0$, but its behavior here is not clear. To understand the behavior near $x = 0$ we will compute the limits as x goes to zero from the left and right.

$$\lim_{x\to 0^+} \frac{x\sqrt{2+\frac{3}{x^2}}}{7x+4} = \lim_{x\to 0^+} \frac{\sqrt{2x^2+3}}{7x+4} = \frac{\sqrt{3}}{4}.$$

$$\lim_{x\to 0^-} \frac{x\sqrt{2+\frac{3}{x^2}}}{7x+4} = \lim_{x\to 0^-} \frac{-\sqrt{2x^2+3}}{7x+4} = \frac{-\sqrt{3}}{4}.$$

We got the negative sign in front of the square root because x is negative. We conclude that the function has a jump discontinuity at $x = 0$. Next we study the behavior for large values of x.

$$\lim_{x\to\infty} \frac{x\sqrt{2+\frac{3}{x^2}}}{7x+4} = \lim_{x\to\infty} \frac{x\sqrt{2+\frac{3}{x^2}}}{7x+4} \frac{\frac{1}{x}}{\frac{1}{x}} =$$

$$\lim_{x\to\infty} \frac{\sqrt{2+\frac{3}{x^2}}}{7+\frac{4}{x}} = \frac{\sqrt{2}}{7}.$$

$$\lim_{x\to-\infty} \frac{x\sqrt{2+\frac{3}{x^2}}}{7x+4} = \lim_{x\to-\infty} \frac{x\sqrt{2+\frac{3}{x^2}}}{7x+4} \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{2+\frac{3}{x^2}}}{7+\frac{4}{x}} = \frac{\sqrt{2}}{7}.$$

Since we didn't have to pull the x inside the square root symbol, the signs are the same.

Lastly we will check to see if the function crosses its horizontal asymptote $y = \frac{\sqrt{7}}{2}$. Set,

$$\frac{x\sqrt{2+\frac{3}{x^2}}}{7x+4} = \frac{\sqrt{2}}{7}$$

and we if we can solve for x. Clearly x cannot be negative or zero so we assume $x > 0$.

$$\sqrt{2x^2+3} = \frac{\sqrt{2}(7x+4)}{7}$$

Squaring both sides gives

$$2x^2+3 = \frac{2(49x^2+56x+16)}{49}.$$

Thus, $2x^2+3 = 2x^2 + \frac{112x+32}{49}$. So, $147 = 112x + 32$ and then $x = \frac{115}{112} \approx 1.0267857$.

We put all the information together to get the graph in Figure 1. $\square$

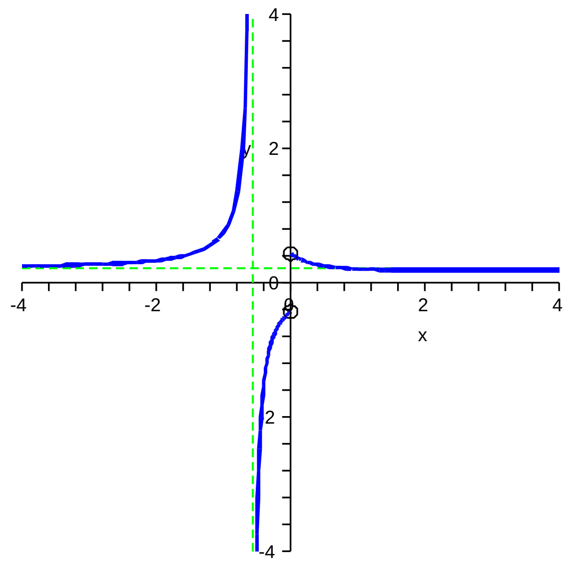

FIGURE 1. Plot of $y = \dfrac{x\sqrt{2+\frac{3}{x^2}}}{7x+4}$.