Limit Examples 2
(L'Hôpital's Rule)

Simply reading these will not help you. You should work them out for yourself, filling in any gaps, looking up facts about functions as needed, and so on. That is, you need to think your way through them.

(1) Find $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x\cos x}$.

Check this has indeterminate form $\frac{0}{0}$. Thus, L'Hôpital's Rule applies.

$$\lim_{x\to 0}\frac{\sin 3x}{x\cos x} = \lim_{x\to 0}\frac{(\sin 3x)'}{(x\cos x)'} = \lim_{x\to 0}\frac{3\cos 3x}{\cos x - x\sin x}$$
$$= \frac{3\cos 0}{\cos 0 - 0} = \frac{3}{1-0} = 3.$$

Or, you could use the "old way".

$$\lim_{x\to 0}\frac{\sin 3x}{x\cos x} = 3\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\lim_{x\to 0}\frac{1}{\cos x} = 3\cdot 1\cdot\frac{1}{1} = 3.$$

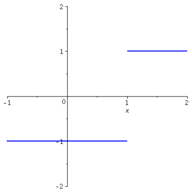
(2) Find $\lim\limits_{y\to 0} \dfrac{1-\cos y}{y}$.

Check this has indeterminate form $\frac{0}{0}$. Thus, L'Hôpital's Rule applies.

$$\lim_{y\to 0}\frac{1-\cos y}{y} = \lim_{y\to 0}\frac{(1-\cos y)'}{(y)'} = \lim_{y\to 0}\frac{\sin y}{1} = 0.$$

(3) Find $\lim\limits_{x\to 1} \dfrac{|x-1|}{x-1}$.

Although this has the indeterminate form $\frac{0}{0}$ we cannot use L'Hôpital's Rule since the numerator is not differentiable. However, we obverse that for $x > 1$ we have $\frac{|x-1|}{x-1} = 1$, while for $x < 1$ we have $\frac{|x-1|}{x-1} = -1$. Therefore, the limit as $x \to 1^+$ is 1, while the limit as $x \to 1^-$ is -1. See graph below. Hence, the limit as $x \to 1$ does not exist.



(4) Find $\lim\limits_{t\to 0} \dfrac{\cos 7t}{\cos 3t}$.

The limit is just $\frac{1}{1} = 1$.

(5) Find $\lim\limits_{t\to 0}\dfrac{\sin 7t}{\sin 3t}$.

Check this has indeterminate form $\frac{0}{0}$. Thus, L'Hôpital's Rule applies.

$$\lim_{t\to 0}\frac{\sin 7t}{\sin 3t}=\lim_{t\to 0}\frac{(\sin 7t)'}{(\sin 3t)'}=\lim_{t\to 0}\frac{7\cos 7t}{3\cos 3t}=\frac{7\cdot 1}{3\cdot 1}=7/3.$$

Or, you could do this one the "old way".

$$\lim_{t\to 0}\frac{\sin 7t}{\sin 3t}=\lim_{t\to 0}\frac{7\sin 7t}{7t}\frac{3t}{3\sin 3t}=\frac{7}{3}\lim_{t\to 0}\frac{\sin 7t}{7t}\cdot\lim_{t\to 0}\frac{3t}{\sin 3t}=\frac{7}{3}\cdot 1\cdot 1=7/3.$$

(6) Find $\lim\limits_{x\to\infty}\dfrac{\arctan(x)-\frac{\pi}{2}}{\ln(1-\frac{1}{x})}$.

This has indeterminate form $\frac{0}{0}$ since $\arctan(x)\to\pi/2$ as $x\to\infty$ and $\ln(1)=0$. Thus, L'Hôpital's Rule applies. The algebra will be a little tricky.

$$\lim_{x\to\infty}\frac{\arctan(x)-\frac{\pi}{2}}{\ln(1-\frac{1}{x})}=\lim_{x\to\infty}\frac{(\arctan(x)-\frac{\pi}{2})'}{(\ln(1-\frac{1}{x}))'}=\lim_{x\to\infty}\frac{\frac{1}{x^2+1}-0}{\frac{1}{1-\frac{1}{x}}\cdot\frac{1}{x^2}}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{x^2+1}}{\frac{1}{x^2-x}}=\lim_{x\to\infty}\frac{x^2-x}{x^2+1}=\lim_{x\to\infty}\frac{x^2-x}{x^2+1}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}=\lim_{x\to\infty}\frac{1-\frac{1}{x}}{1+\frac{1}{x^2}}=\frac{1-0}{1+0}=1.$$

(7) Find $\lim\limits_{z\to 0}\dfrac{\sin 2z}{\sinh 3z}$.

Check this has indeterminate form $\frac{0}{0}$. Thus, L'Hôpital's Rule applies.

$$\lim_{z\to 0}\frac{\sin 2z}{\sinh 3z}=\lim_{z\to 0}\frac{(\sin 2z)'}{(\sinh 3z)'}=\lim_{z\to 0}\frac{2\cos 2z}{3\cosh 3z}=\frac{2\cos 0}{3\cosh 0}=2/3.$$

(8) Find $\lim\limits_{x\to 1}(x-1)\tan(\pi x/2)$.

Check this has indeterminate form $0\cdot(\pm\infty)$. In order to apply L'Hôpital's Rule we rewrite this as $\lim\limits_{x\to 1}\dfrac{x-1}{\cot(\pi x/2)}$. This has indeterminate form $\frac{0}{0}$. Then

$$\lim_{x\to 1}\frac{x-1}{\cot(\pi x/2)}=\lim_{x\to 1}\frac{(x-1)'}{(\cot(\pi x/2))'}=\lim_{x\to 1}\frac{1}{-\pi/2\cdot\csc^2(\pi x/2)}=$$

$$-\frac{2}{\pi}\lim_{x\to 1}\sin^2(\pi x/2)=-\frac{2}{\pi}\cdot\sin^2(\pi/2)=-\frac{2}{\pi}.$$

(9) Find $\lim\limits_{x\to 2^+}\dfrac{\ln(x-2)}{\cot\pi x}$.

Check this has indeterminate form $\frac{-\infty}{\infty}$. Thus, L'Hôpital's Rule applies.

$$\lim_{x\to 2^+} \frac{\ln(x-2)}{\cot \pi x} = \lim_{x\to 2^+} \frac{(\ln(x-2))'}{(\cot \pi x)'} = \lim_{x\to 2^+} \frac{\frac{1}{x-2}}{-\pi \csc^2 \pi x} = -\frac{1}{\pi} \lim_{x\to 2^+} \frac{\sin^2 \pi x}{x-2}.$$

This has indeterminate form $\frac{0}{0}$. Thus, we may apply L'Hôpital's Rule again.

$$-\frac{1}{\pi} \lim_{x\to 2^+} \frac{(\sin^2 \pi x)'}{(x-2)'} = -\frac{1}{\pi} \lim_{x\to 2^+} \frac{2\pi \sin \pi x \cos \pi x}{1} = -2\cdot 0\cdot 1 = 0.$$

(10) Find $\lim_{x\to 3} \frac{x^2-3x+1}{x^2-6x+8}$.

When we plug in $x=3$ we get $\frac{1}{-1} = -1$. If you tried to use L'Hôpital's Rule you would get the wrong answer.

(11) Find $\lim_{p\to 0} \frac{\ln \sec p}{p^2}$.

Check this has indeterminate form $\frac{0}{0}$, since $\ln \sec 0 = \ln 1 = 0$. Thus, L'Hôpital's Rule applies.

$$\lim_{p\to 0} \frac{\ln \sec p}{p^2} = \lim_{p\to 0} \frac{(\ln \sec p)'}{(p^2)'} = \lim_{p\to 0} \frac{\frac{1}{\sec p}\sec p \tan p}{2p} =$$

$$\lim_{p\to 0} \frac{\tan p}{2p} = \frac{1}{2} \lim_{p\to 0} \frac{\sin p}{p}\frac{1}{\cos p} = \frac{1}{2}\cdot 1\cdot 1 = 1/2.$$

(12) Find $\lim_{x\to \frac{\pi}{2}} \frac{\cos 3x}{\cos x}$.

Check this has indeterminate form $\frac{0}{0}$, since $\cos \frac{3\pi}{2}$ and $\cos \frac{\pi}{2}$ are both 0. Thus, L'Hôpital's Rule applies.

$$\lim_{x\to \frac{\pi}{2}} \frac{\cos 3x}{\cos x} = \lim_{x\to \frac{\pi}{2}} \frac{-3\sin 3x}{-\sin x} = 3\cdot \frac{\sin \frac{3\pi}{2}}{\sin \frac{\pi}{2}} = 3\cdot \frac{-1}{1} = -3.$$

Or, you could use the formula $\cos(a+b) = \cos a \cos b - \sin a \sin b$, twice, to show that $\cos 3x = -3\cos x + 4\cos^3 x$. Then the limit becomes

$$\lim_{x\to \frac{\pi}{2}} \frac{-3\cos x + 4\cos^3 x}{\cos x} = \lim_{x\to \frac{\pi}{2}} -3 + 4\cos^2 x = -3 + 4\cos \frac{\pi}{2} = -3 + 4\cdot 0 = -3.$$

(13) Find $\lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin x}$.

This has indeterminate form $\infty - \infty$. The following algebraic trick will be used.

$$a-b=\frac{1}{\frac{1}{a}}-\frac{1}{\frac{1}{b}}=\frac{\frac{1}{b}-\frac{1}{a}}{\frac{1}{ab}}.$$

Thus,

$$\lim_{x\to 0}\frac{1}{x}-\frac{1}{\sin x}=\lim_{x\to 0}\frac{\sin x-x}{x\sin x}.$$

This has indeterminate form $\frac{0}{0}$. We apply L'Hôpital's Rule.

$$\lim_{x\to 0}\frac{\sin x-x}{x\sin x}=\lim_{x\to 0}\frac{(\sin x-x)'}{(x\sin x)'}=\lim_{x\to 0}\frac{\cos x-1}{\sin x+x\cos x}.$$

This still has indeterminate form $\frac{0}{0}$. We apply L'Hôpital's Rule again.

$$\lim_{x\to 0}\frac{(\cos x-1)'}{(\sin x+x\cos x)'}=\lim_{x\to 0}\frac{-\sin x}{2\cos x-x\sin x}=\frac{-0}{2-0}=0.$$

(14) Find $\lim\limits_{x\to 0}\frac{1}{e^x-1}-\frac{1}{x}$.

This has indeterminate form $\infty-\infty$. We use the same algebraic trick as above. Thus,

$$\lim_{x\to 0}\frac{1}{e^x-1}-\frac{1}{x}=\lim_{x\to 0}\frac{x-(e^x-1)}{x(e^x-1)}.$$

This has indeterminate form $\frac{0}{0}$. We apply L'Hôpital's Rule.

$$\lim_{x\to 0}\frac{(x-(e^x-1))'}{(x(e^x-1))'}=\lim_{x\to 0}\frac{1-e^x}{e^x-1+xe^x}.$$

This still has indeterminate form $\frac{0}{0}$. We apply L'Hôpital's Rule again.

$$\lim_{x\to 0}\frac{(1-e^x)'}{(e^x-1+xe^x)'}=\lim_{x\to 0}\frac{-e^x}{2e^x+xe^x}=\frac{-e^0}{2e^0+0}=-\frac{1}{2}$$

(15) Find $\lim\limits_{x\to 0^+}x^x$. Check that this has indeterminate form 0^0. Let $L=\lim\limits_{x\to 0^+}x^x$. Then

$$\ln L=\ln\lim_{x\to 0^+}x^x=\lim_{x\to 0^+}\ln x^x=\lim_{x\to 0^+}x\ln x.$$

This has indeterminate form $0(-\infty)$. We can rewrite it as

$$\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}.$$

Now it has indeterminate form $\frac{-\infty}{\infty}$. We apply L'Hôpital's Rule.

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{(\ln x)'}{(\frac{1}{x})'}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-1}{x^2}}=\lim_{x\to 0^+}-x=0.$$

Thus, $\ln L=0$, so $L=1$.

(16) Find $\lim\limits_{x\to\frac{\pi}{2}^-}(\tan x)^{(\cos x)}$. Check that the indeterminate form is ∞^0. Let $L = \lim\limits_{x\to\frac{\pi}{2}^-}(\tan x)^{(\cos x)}$. Then

$$\ln L = \ln \lim_{x\to\frac{\pi}{2}^-}(\tan x)^{(\cos x)} = \lim_{x\to\frac{\pi}{2}^-}\ln(\tan x)^{(\cos x)} = \lim_{x\to\frac{\pi}{2}^-}(\cos x)\ln(\tan x).$$

This has indeterminate form $0\cdot\infty$. But, we can rewrite this last limit as

$$\lim_{x\to\frac{\pi}{2}^-}(\cos x)\ln(\tan x) = \lim_{x\to\frac{\pi}{2}^-}\frac{\ln(\tan x)}{\sec x}.$$

This has indeterminate form $\frac{\infty}{\infty}$. Now we can apply L'Hôpital's Rule.

$$\lim_{x\to\frac{\pi}{2}^-}\frac{\ln(\tan x)}{\sec x} = \lim_{x\to\frac{\pi}{2}^-}\frac{(\ln(\tan x))'}{(\sec x)'} = \lim_{x\to\frac{\pi}{2}^-}\frac{\frac{\sec^2 x}{\tan x}}{\sec x\tan x}$$

$$= \lim_{x\to\frac{\pi}{2}^-}\frac{\sec x}{\tan^2 x}\cdot\frac{\cos x}{\cos x} = \lim_{x\to\frac{\pi}{2}^-}\frac{1}{\frac{\sin^2 x}{\cos x}}$$

$$= \lim_{x\to\frac{\pi}{2}^-}\frac{\cos x}{\sin^2 x} = \frac{0}{1^2} = 0.$$

Thus, $\ln L = 0$, so $L = 1$.

(17) Find $\lim\limits_{x\to 0}(1+3x)^{\frac{1}{2x}}$.

Method I: Check that the indeterminate form is 1^∞. Let $L = \lim\limits_{x\to 0}(1+3x)^{\frac{1}{2x}}$. Then

$$\ln L = \ln\lim_{x\to 0}(1+3x)^{\frac{1}{2x}} = \lim_{x\to 0}\ln(1+3x)^{\frac{1}{2x}} = \lim_{x\to 0}\frac{1}{2x}\ln(1+3x) = \lim_{x\to 0}\frac{\ln(1+3x)}{2x}.$$

Now, the indeterminate form is $\frac{0}{0}$ and we may apply L'Hôpital's Rule.

$$\lim_{x\to 0}\frac{\ln(1+3x)}{2x} = \lim_{x\to 0}\frac{(\ln(1+3x))'}{(2x)'} = \lim_{x\to 0}\frac{\frac{3}{1+3x}}{2} = \frac{\frac{3}{1+0}}{2} = 3/2.$$

This, $\ln L = 3/2$, so $L = e^{\frac{3}{2}}$.

Method II: Recall that by definition $e = \lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}$. Let $y = 3x$ and note that as $x\to 0$ we have that $y\to 0$. Therefore,

$$\lim_{x\to 0}(1+3x)^{\frac{1}{2x}} = \lim_{y\to 0}(1+y)^{\frac{3}{2y}} = \left(\lim_{y\to 0}(1+y)^{\frac{1}{y}}\right)^{\frac{3}{2}} = e^{\frac{3}{2}}.$$

(18) Find $\lim_{u\to 0} \coth u - \cot u$.

Check this has indeterminate form $\infty - \infty$. You can show that

$$\lim_{u\to 0} \coth u - \cot u = \lim_{u\to 0} \frac{\tan u - \tanh u}{\tan u \tanh u}.$$

This has indeterminate form $\frac{0}{0}$. We apply L'Hôpital's Rule.

$$\lim_{u\to 0} \frac{\tan u - \tanh u}{\tan u \tanh u} = \lim_{u\to 0} \frac{(\tan u - \tanh u)'}{(\tan u \tanh u)'} = \lim_{u\to 0} \frac{\sec^2 u - \operatorname{sech}^2 u}{\sec^2 u \tanh u + \tan u \operatorname{sech}^2 u}.$$

This still has indeterminate form $\frac{0}{0}$. Applying L'Hôpital's Rule now would be very messy. Instead we do the following.

$$\lim_{u\to 0} \frac{\sec^2 u - \operatorname{sech}^2 u}{\sec^2 u \tanh u + \tan u \operatorname{sech}^2 u} \cdot \frac{\cos^2 u \cosh^2 u}{\cos^2 u \cosh^2 u} = \lim_{u\to 0} \frac{\cosh^2 u - \cos^2 u}{\sinh u \cosh u + \sin u \cos u}.$$

This still has indeterminate form $\frac{0}{0}$. Now we apply L'Hôpital's Rule.

$$\lim_{u\to 0} \frac{\cosh^2 u - \cos^2 u}{\sinh u \cosh u + \sin u \cos u} = \lim_{u\to 0} \frac{(\cosh^2 u - \cos^2 u)'}{(\sinh u \cosh u + \sin u \cos u)'} =$$

$$\lim_{u\to 0} \frac{2\cosh u \sinh u + 2\cos u \sin u}{\cosh^2 u + \sinh^2 u + \cos^2 u - \sin^2 u} = \frac{0}{2} = 0.$$