Some Notation From Set Theory for Calculus Students

A **set** is a collection of **elements**. The expression "$p \in S$" means p is an element of the set S. A set may be defined in several ways: in ordinary English, *e.g.,* let A be the set of positive even integers; by listing its elements within braces, *e.g.,* let $A = \{2, 4, 6, 8, ...\}$; or by using "set builder" notation, *e.g.,* $A = \{n \in \mathbb{Z} \,|\, n > 0 \,\text{and}\, n \,\text{is even}\,\}$, read, A is the set of all integers n such that $n > 0$ and n is even ($\mathbb{Z}$ is the standard notation for the integers).

A set does not have an order. Thus $\{a, b\} = \{b, a\}$. An **ordered set** is a set together with an ordering. When we want to stress that a set has been endowed with an ordering we will use parenthesizes instead of braces: (a, b) is an ordered set and is not equal to (b, a).

The following notations are standard.

- $\phi = \{\}$, the empty set.
- $A \subset B$: read A is a subset of B, means every element of A is an element of B. *Example:* $\{2, 5\} \subset \{1, 2, 3, 4, 5\}$.
- $A \cup B$: read A union B, means the set of all elements that are in A **or** in B. *Example:* $\{\$, *, !\} \cup \{\alpha, !, \star, 17\} = \{\$, *, !, \alpha, \star, 17\}$.
- $A \cap B$: read A intersection B, means the set of all elements that are in A **and** in B. *Example:* $\{\$, *, !\} \cap \{\alpha, !, \star, 17\} = \{!\}$.
- $A - B$: read A minus B, means the set of all elements of A that are not elements of B. *Example:* $\{\$, *, !\} - \{\alpha, !, \star, 17\} = \{\$, *\}$.
- $A \times B$: read A cross B, means the set of ordered pairs (a, b) where $a \in A$ and $b \in B$. Since there is a natural one-to-one correspondence between $(A \times B) \times C$ and $A \times (B \times C)$, $((a, b), c) \longleftrightarrow (a, (b, c))$, we shall ignore the distinction between them and use the notation $A \times B \times C$ for the set $\{(a, b, c) \,|\, a \in A, b \in B, \,\text{and}\, c \in C\}$. Other multiple cross products are defined similarly. *Examples:* $\{1, 3\} \times \{0, 1, 2\} = \{(1, 0), (1, 1), (1, 2), (3, 0), (3, 1), (3, 2)\}$. $\{*, \#\} \times \{\%\} = \{(*, \%), (\#, \%)\}$.
- $A^n = A \times \cdots \times A$, n times. *Example:* $\{2, 3\}^3 = \{(2, 2, 2), (2, 2, 3), (2, 3, 2), (2, 3, 3), (3, 2, 2), (3, 2, 3), (3, 3, 2), (3, 3, 3)\}$.

Some standard sets are:

- $\mathbb{Z}$: the integers (most likely from the German *Zahl*, meaning number),
- $\mathbb{Q}$: the rational numbers (quotients),
- $\mathbb{R}$: the real numbers, and
- $\mathbb{C}$: the complex numbers.

Remark: The sets $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are normally given an ordering. Interestingly, $\mathbb{C}$ is not typically ordered.

Interval Notation.

$$\begin{array}{rcl@{\qquad}rcl}
[a, b] & = & \{x \in \mathbb{R} \,|\, a \leq x \leq b\} & [a, \infty) & = & \{x \in \mathbb{R} \,|\, a \leq x\} \\
(a, b) & = & \{x \in \mathbb{R} \,|\, a < x < b\} & (a, \infty) & = & \{x \in \mathbb{R} \,|\, a < x\} \\
(a, b] & = & \{x \in \mathbb{R} \,|\, a < x \leq b\} & (-\infty, b] & = & \{x \in \mathbb{R} \,|\, x \leq b\} \\
[a, b) & = & \{x \in \mathbb{R} \,|\, a \leq x < b\} & (-\infty, b) & = & \{x \in \mathbb{R} \,|\, x < b\}
\end{array}$$

Remark: The notation "(a, b)" is ambiguous; it could represent an interval or an ordered pair. One has to consider the context to understand the intended meaning. On behalf of mathematicians everywhere I apologize for any in convenience this may cause.

Examples.

- $(-\infty, -\sqrt{7}\,] \cup [\sqrt{7}, \infty) = \{x \in \mathbb{R} \,|\, x \le -\sqrt{7}\} \cup \{x \in \mathbb{R} \,|\, x \ge \sqrt{7}\}$ is the solution set for $x^2 - 7 \ge 0$.
- $(-\infty, 0) \cup (0, \infty) = \mathbb{R} - \{0\}$ is the natural domain of $1/x$.
- $\mathbb{R}^2$ is the plane. $\mathbb{R}^3$ is 3-dimensional space. $\mathbb{R}^4$ is 4-dimensional space. And so on.
- $\phi \subset A$, $\phi = A \cap \phi$, and $A = A \cup \phi$ are true statements for all sets A.
- $\{x \in \mathbb{R} \,|\, -2 \le x < 5\} = [-2, 5) = [-2, 7] \cap (-10, 5)$.
- $S = [0, 1] \times [0, 1]$ is the *unit square* in the plane $\mathbb{R}^2$ with corners (0,0), (1,0), (0,1), and (1,1).

Quantifiers. The symbols $\forall$ and $\exists$ are rather handy. $\forall$ means "for all." $\exists$, means "there exists." They are called *quantifiers* and are commonly used in logic.

Examples.

- $\forall x \ge 0\ \exists y \ge 0$ such that $y^2 = x$. This means, every nonnegative real number has a nonnegative square root.
- A function f has a *relative maximum* at c if $\exists \epsilon > 0$ such that $\forall x \in (c - \epsilon, c + \epsilon)$ we have $f(x) \le f(c)$.
- A function f is *unbounded from above* if $\forall B > 0\ \exists x \in \mathbb{R}$ such that $f(x) > B$.

Problems.

1. Describe $[0, 1] \times [0, 2] \times [0, 3]$.
2. Simplify $((1, 3) \cap (2, 5)) \cup [3, 4)$.
3. Let $A = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \le 9\}$, $B = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \le 4\}$, and $C = \{(x, y) \in \mathbb{R}^2 \,|\, y \ge 0\}$. Draw $A - B$, $A - C$, $A \cap C$, $(A - B) \cap C$, and $A - (B \cap C)$.
4. Find the solution set in $\mathbb{R}^2$ of $\sin x \cos y = 0$.
5. Draw $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times \mathbb{R}$, and $((0, 1] \cup \{2, 3\}) \times ([-2, -1] \cup (2, 3))$ as subsets of $\mathbb{R}^2$.
6. Let A be a set. What is $A \times \phi$?
7. Let A, B, and C be sets. Prove that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. (You can draw pictures to "see" this, but you need to reason from the definitions to prove it.)
8. (a) Write a definition for a point to be a relative minimum of a function using quantifiers.
   (b) Write a definition for a function to be unbounded from below using quantifiers.
   (c) Translate "$\forall\ \delta > 0\ \exists\ N \in \mathbb{Z}$ such that $\forall$ integers $n > N$ we have $0 < \frac{1}{n} < \delta$" into English. Is it a true statement?