Derivation of the formulas for cosh(x) and sinh(x)[1]

Let u be the area of the region OAB in Figure 1. Here O is the origin (0,0). The curve is the unit hyperbola $x^2 - y^2 = 1$. The points A and B have the same x coordinates and are the same distance from the x-axis. Let their coordinates be denoted $(c(u), s(u))$ and $(c(u), -s(u))$, respectively. We think of c and s as functions determined by the value of u. Thus, $c(0) = 1$, $s(0) = 0$, and both have infinite limits as $u \to \infty$. But, we want to find formulas for them. Then we are justified in defining $\cosh(u)$ and $\sinh(u)$ by these formulas.

We rotate Figure 1 by 45^o counterclockwise to get Figure 2. Let A' and B' be the images of A and B, respectively. If we denote the coordinates of A' by $(\alpha(u), \beta(u))$, then the coordinates of B' are $(\beta(u), \alpha(u))$ by symmetry through the line $x = y$.

In the Appendix we show that the image of the curve has equation

$$xy = \frac{1}{2},$$

and that

$$c = \frac{\beta + \alpha}{\sqrt{2}} \qquad \& \qquad s = \frac{\beta - \alpha}{\sqrt{2}}.$$

Thus, if we can find formulas for α and β in terms of u we are essentially done.

We drop perpendicular lines from A' and B' to points C and D, respectively, on the x-axis. The coordinates of C are $(\alpha(u), 0)$, while the coordinates of D are $(\beta(u), 0)$. See Figure 3.

Let N be the area under the graph of $xy = 1/2$ from $x = C$ to $x = D$. Let T_1 be the area of the triangle OCA', and let T_2 be the area of the triangle ODB'. We observe that

$$u = N + T_1 - T_2.$$

Notice that adding T_1 adds on the area of the small triangle with base OC, but the subtracting T_2 cancels this out. But $T_1 = T_2 = \dfrac{\alpha\beta}{2}$. Thus we have

$$u = N = \int_{\alpha}^{\beta} \frac{1}{2x}\, dx.$$

Now we solve for α and β in terms of u. Integration gives

$$\ln\beta - \ln\alpha = 2u. \qquad (1)$$

Notice $xy = 1/2$ implies $\alpha\beta = 1/2$. Hence

$$\ln\alpha + \ln\beta = \ln\frac{1}{2}. \qquad (2)$$

We add equations (1) and (2) then solve for $\ln\beta$, to get

$$\ln\beta = \frac{2u + \ln\frac{1}{2}}{2} = u + \ln\frac{1}{\sqrt{2}}.$$

Applying the exponential function to both sides gives

$$\beta = e^{(u + \ln\frac{1}{\sqrt{2}})} = e^u e^{\ln\frac{1}{\sqrt{2}}} = \frac{e^u}{\sqrt{2}}.$$

By subtracting equations (1) and (2) we can also show $\alpha = \dfrac{e^{-u}}{\sqrt{2}}$. Thus we get

$$\cosh(u) = c(u) = \frac{e^u + e^{-u}}{2} \qquad \& \qquad \sinh(u) = s(u) = \frac{e^u - e^{-u}}{2}.$$

Appendix on Rotations

Suppose we want to rotate a graph in the xy-plane by θ degrees counterclockwise. First consider where the point $(1, 0)$ would land. We can use some trig to see that the new coordinates are $(\cos\theta, \sin\theta)$. It also easy to see that $(0, 1)$ gets rotated to $(-\sin\theta, \cos\theta)$. See Figure 4.

Now select an arbitrary point (x, y) and denote the point it gets rotate to as (a, b). We can think of the points in the plane as vectors. Then since $(x, y) = x(1, 0) + y(0, 1)$ we get that

$$a = x\cos\theta - y\sin\theta$$

$$b = x\sin\theta + y\cos\theta$$

Now consider the graph of $x^2 - y^2 = 1$ $(x \geq 1)$. We want to rotate it by $\pi/4$ and find an equation that describes the resulting graph. Another way of saying this is, given a pair (a, b) we want to find an equation in a and b that will tell us if (a, b) is a point on the rotated hyperbola. To do this we shall start with (a, b) rotate it by $-\pi/4$, call this point (x, y) and then plug into $x^2 - y^2 = 1$. Thus,

$$x = \frac{\sqrt{2}}{2}a + \frac{\sqrt{2}}{2}b$$

$$y = -\frac{\sqrt{2}}{2}a + \frac{\sqrt{2}}{2}b$$

Substitution gives

$$\left(\frac{\sqrt{2}}{2}a + \frac{\sqrt{2}}{2}b\right)^2 - \left(-\frac{\sqrt{2}}{2}a + \frac{\sqrt{2}}{2}b\right)^2 = 1$$

This simplifies to

$$ab = \frac{1}{2}$$

as the reader can check. Finally, $(\cosh(u), \sinh(u))$ is obtained by rotating (α, β) by $-\pi/4$. Hence $\cosh(u) = (\beta + \alpha)/\sqrt{2}$ and $\sinh(u) = (\beta - \alpha)/\sqrt{2}$ as claimed.

Figures

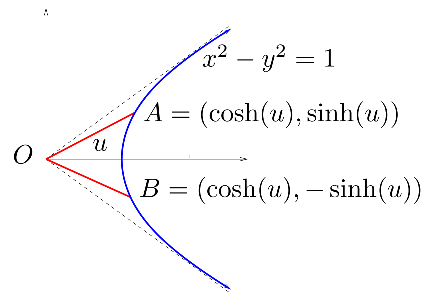

Figure 1: Definitions of hyperbolic sine and cosine

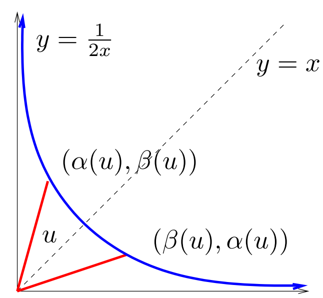

Figure 2: Rotation by $\pi/4$

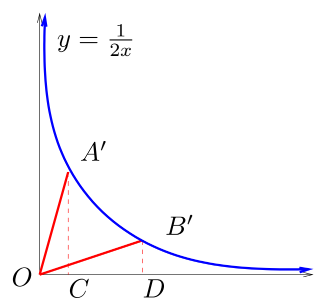

Figure 3: Comparing areas

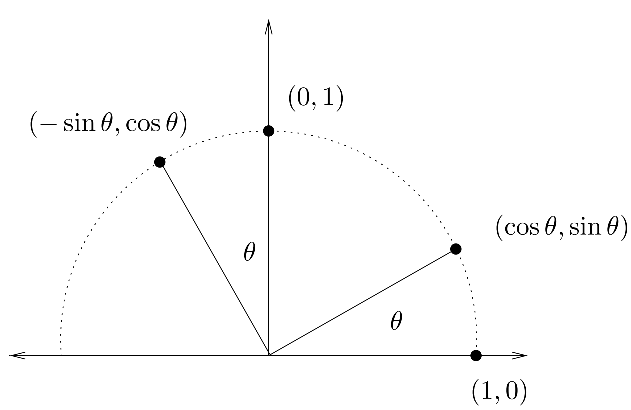

Figure 4: Rotating the Plane.