Limit Examples

(1) $\lim_{\theta\to 0}\theta\csc\theta = \lim_{\theta\to 0}\frac{\theta}{\sin\theta} = \frac{1}{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}} = \frac{1}{1} = 1.$

(2) $\lim_{\theta\to 0}\frac{\tan 2\theta\sin^2\theta}{\theta^3} = \lim_{\theta\to 0}\frac{2}{2}\frac{\sin 2\theta}{\theta\cos 2\theta}\frac{\sin\theta}{\theta}\frac{\sin\theta}{\theta} = 2\lim_{\theta\to 0}\frac{\sin 2\theta}{2\theta}\frac{1}{\cos 2\theta}\frac{\sin\theta}{\theta}\frac{\sin\theta}{\theta} =$ $2\cdot 1\cdot\frac{1}{1}\cdot 1\cdot 1 = 2.$

(3) $\lim_{\theta\to 0}\frac{\cos 3\theta}{\theta} =$ "$\frac{1}{0}$" is undefined. (The left and right limits are $-\infty$ and ∞, respectively.)

(4) $\lim_{\theta\to\pi/2}\frac{\cos\theta}{\theta-\pi/2} = ?$ Recall $\cos\theta = -\sin\left(\theta-\frac{\pi}{2}\right)$. Let $q = \theta-\frac{\pi}{2}$.

Thus, $? = \lim_{\theta\to\pi/2}\frac{-\sin\left(\theta-\frac{\pi}{2}\right)}{\theta-\frac{\pi}{2}} = \lim_{q\to 0}\frac{-\sin q}{q} = -1.$

(5) $\lim_{\theta\to 0}\frac{\theta\sec\theta}{\csc\theta} = \lim_{\theta\to 0}\frac{\theta\sin\theta}{\cos\theta} = \frac{0\cdot 0}{1} = 0.$

(6) $\lim_{\theta\to 0}\frac{\sin 3\theta}{\sin\theta} = \lim_{\theta\to 0}\frac{3}{3}\frac{\theta}{\theta}\frac{\sin 3\theta}{\sin\theta} = 3\lim_{\theta\to 0}\frac{\sin 3\theta}{3\theta}\frac{\theta}{\sin\theta} = 3\cdot 1\cdot 1 = 3.$

(7) $\lim_{\theta\to 0}\frac{\tan\theta}{\sin 2\theta} = \lim_{\theta\to 0}\frac{\tan\theta}{2\sin\theta\cos\theta} = \frac{1}{2}\lim_{\theta\to 0}\frac{1}{\cos^2\theta} = \frac{1}{2}\cdot\frac{1}{1^2} = \frac{1}{2}.$

(8) $\lim_{\theta\to 0}\frac{\theta}{\theta+\tan\theta} = \lim_{\theta\to 0}\frac{\theta}{\theta+\tan\theta}\frac{\frac{1}{\theta}}{\frac{1}{\theta}} = \lim_{\theta\to 0}\frac{1}{1+\frac{\tan\theta}{\theta}} = \frac{1}{1+1} = \frac{1}{2}.$

(9) $\lim_{x\to\infty}\cos\arctan x = \cos\frac{\pi}{2} = 0.$

(10) $\lim_{x\to\infty}\frac{\tan x}{x}$ is undefined. Why?

(11) $\lim_{x\to\infty}\frac{\frac{1}{x}+\sin\theta}{\cos\theta} = \frac{0+\sin\theta}{\cos\theta} = \tan\theta.$ (Since θ does not depend on x.)

(12) $\lim_{x\to\pi^+}\frac{x\sin x+\csc x}{3x} = \lim_{x\to\pi^+}\frac{\sin x}{3}+\frac{\csc x}{3x} = \frac{0}{3}+\frac{-\infty}{3\pi} = -\infty.$

(13) $\lim_{y\to-\pi/2^-}\frac{\cot y-\tan y}{y} = \frac{0-\infty}{-\pi/2} = \infty.$

(14) $\lim_{x\to\infty} \frac{x+4}{x^2-16} = 0.$

(15) $\lim_{x\to 4} \frac{x+4}{x^2-16} = \lim_{x\to 4} \frac{1}{x-4}$ is undefined. (The left and right limits are $-\infty$ and ∞, respectively.)

(16) $\lim_{x\to -4} \frac{x+4}{x^2-16} = \lim_{x\to -4} \frac{1}{x-4} = -\frac{1}{8}.$

(17) $\lim_{x\to 0} \frac{x+4}{x^2-16} = \frac{0+4}{0^2-16} = -\frac{1}{4}.$

(18) $\lim_{x\to\infty} \frac{4e^x+8}{3e^x+7} = \lim_{x\to\infty} \frac{4e^x+8}{3e^x+7}\frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{4+8e^{-x}}{3+7e^{-x}} = \frac{4+0}{3+0} = \frac{4}{3}.$

(19) $\lim_{x\to -\infty} \frac{4e^x+8}{3e^x+7} = \frac{0+8}{0+7} = \frac{8}{7}.$

(20) $\lim_{x\to\infty} \arctan \frac{x^2+x+1}{x^2+3x+4} = \arctan \lim_{x\to\infty} \frac{x^2+x+1}{x^2+3x+4} \arctan 1 = \frac{\pi}{4}.$

(21) $\lim_{x\to\infty} \frac{\sqrt{6x^3+x}}{x^2+2} = \lim_{x\to\infty} \sqrt{\left(\frac{\sqrt{6x^3+x}}{x^2+2}\right)^2} = \sqrt{\lim_{x\to\infty} \frac{6x^3+x}{x^4+4x+4}} = \sqrt{0} = 0.$

(22) $\lim_{x\to\infty} \frac{x^2+x+7}{\sqrt{x^4+8}} = \sqrt{\lim_{x\to\infty} \frac{x^4+2x^3+15x^2+14x+49}{x^4+8}} = \sqrt{1} = 1.$

Or, $\lim_{x\to\infty} \frac{x^2+x+7}{\sqrt{x^4+8}} = \lim_{x\to\infty} \frac{x^2+x+7}{\sqrt{x^4+8}}\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1+\frac{1}{x}+\frac{7}{x^2}}{\sqrt{1+\frac{8}{x^4}}} = \frac{1}{\sqrt{1}} = 1.$

(23) $\lim_{x\to\infty} \sqrt{17x^2+7} - 4x = \lim_{x\to\infty} \frac{\sqrt{17x^2+7}-4x}{1}\frac{\sqrt{17x^2+7}+4x}{\sqrt{17x^2+7}+4x} =$

$\lim_{x\to\infty} \frac{17x^2+7-16x^2}{\sqrt{17x^2+7}+4x} = \lim_{x\to\infty} \frac{x^2+7}{\sqrt{17x^2+7}+4x}\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x+\frac{7}{x}}{\sqrt{17+\frac{7}{x}}+4} =$

$\lim_{x\to\infty} \frac{\infty+0}{\sqrt{17+0}+4} = \infty.$

(24) $\lim_{x\to\infty} \sqrt{16x^2+7} - 4x = \lim_{x\to\infty} \frac{\sqrt{16x^2+7} - 4x}{1} \frac{\sqrt{16x^2+7}+4x}{\sqrt{16x^2+7}+4x} =$

$\lim_{x\to\infty} \frac{16x^2+7-16x^2}{\sqrt{16x^2+7}+4x} = \lim_{x\to\infty} \frac{7}{\sqrt{16x^2+7}+4x} \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{7}{x}}{\sqrt{16+\frac{7}{x}}+4} =$

$\lim_{x\to\infty} \frac{0}{\sqrt{16+0}+4} = 0.$

(25) $\lim_{x\to\infty} \sqrt{16x^2+x} - 4x = \lim_{x\to\infty} \frac{\sqrt{16x^2+x} - 4x}{1} \frac{\sqrt{16x^2+x}+4x}{\sqrt{16x^2+x}+4x} =$

$\lim_{x\to\infty} \frac{16x^2+x-16x^2}{\sqrt{16x^2+x}+4x} = \lim_{x\to\infty} \frac{x}{\sqrt{16x^2+x}+4x} \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{16+\frac{1}{x}}+4} =$

$\lim_{x\to\infty} \frac{1}{\sqrt{16+0}+4} = \frac{1}{8}.$

An Application. Graph $y = \frac{x\sqrt{2+\frac{3}{x^2}}}{7x+4}$.

Solution. First we observe that there is a vertical asymptote at $x = -4/7$. Also notice that the function is undefine at $x = 0$, but its behavior here is not clear. To understand the behavior near $x = 0$ we will compute the limits as x goes to zero from the left and right.

$$\lim_{x\to 0^+} \frac{x\sqrt{2+\frac{3}{x^2}}}{7x+4} = \lim_{x\to 0^+} \frac{\sqrt{2x^2+3}}{7x+4} = \frac{\sqrt{3}}{4}.$$

$$\lim_{x\to 0^-} \frac{x\sqrt{2+\frac{3}{x^2}}}{7x+4} = \lim_{x\to 0^-} \frac{-\sqrt{2x^2+3}}{7x+4} = \frac{-\sqrt{3}}{4}.$$

We got the negative sign in front of the squar root because x is negative. We conclude that the function has a jump discontinuity at $x = 0$. Next we study the behavior for large values of x.

$$\lim_{x\to\infty} \frac{x\sqrt{2+\frac{3}{x^2}}}{7x+4} = \lim_{x\to\infty} \frac{x\sqrt{2+\frac{3}{x^2}}}{7x+4} \frac{\frac{1}{x}}{\frac{1}{x}} =$$

$$\lim_{x\to\infty} \frac{\sqrt{2+\frac{3}{x^2}}}{7+\frac{4}{x}} = \frac{\sqrt{2}}{7}.$$

$$\lim_{x\to -\infty} \frac{x\sqrt{2+\frac{3}{x^2}}}{7x+4} = \lim_{x\to -\infty} \frac{x\sqrt{2+\frac{3}{x^2}}}{7x+4} \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x\to -\infty} \frac{\sqrt{2+\frac{3}{x^2}}}{7+\frac{4}{x}} = \frac{\sqrt{2}}{7}.$$

Since we didn't have to pull the x inside the square root symbol, the signs are the same.

Lastly we will check to see if the function crosses its horizontal asymtote $y = \frac{\sqrt{7}}{2}$. Set,

$$\frac{x\sqrt{2+\frac{3}{x^2}}}{7x+4} = \frac{\sqrt{2}}{7}$$

and we if we can solve for x. Clearly x cannot be negative or zero so we assume $x > 0$.

$$\sqrt{2x^2+3} = \frac{\sqrt{2}(7x+4)}{7}$$

Squaring both sides gives

$$2x^2+3 = \frac{2(49x^2+56x+16)}{49}.$$

Thus, $2x^2+3 = 2x^2 + \frac{112x+32}{49}$. So, $147 = 112x + 32$ and then $x = \frac{115}{112} \approx 1.0267857$.

We put all the information together to get the graph in Figure 1. □

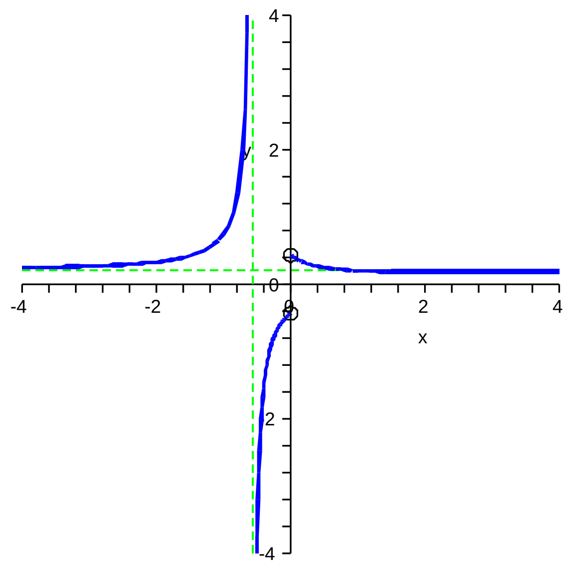

FIGURE 1. Plot of $y = \dfrac{x\sqrt{2+\frac{3}{x^2}}}{7x+4}$.