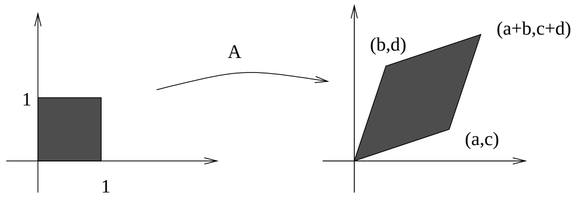
Determinants[1]

1. Introduction

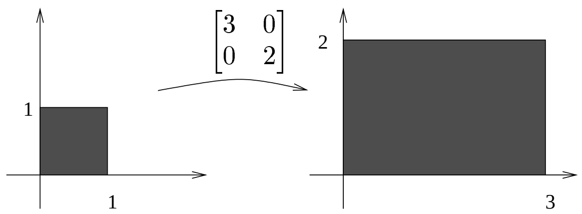
Definition 1. For a 2×2 matrix $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$. We also use the notation, $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

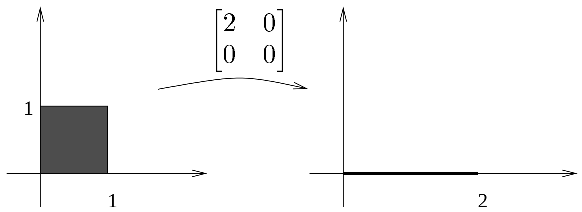
Think of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as a map from R^2 to R^2. Let $S = [0,1] \times [0,1]$ be the unit square:

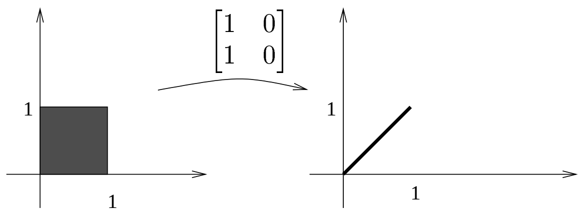


Then the image of S under A is a parallelogram with area $|\det A|$.

Examples:







Homework Problem 1. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let $S = [0,1] \times [0,1]$ be the unit square. Prove the claim made above that $A(S)$ has area $|\det(A)|$.

Notice that, intuitively, the area of $A(S) = 0$ implies $A : R^2 \to R^2$ is not one-to-one, implies A is not invertible.

Homework Problem 2. Again let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Prove that A^{-1} exists if and only if $\det(A) \neq 0$.

Homework Problem 3. Prove that for 2×2 matrices $\det A \det B = \det AB$.

Definition 2.

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - d \begin{vmatrix} b & b \\ h & i \end{vmatrix} + g \begin{vmatrix} b & c \\ e & f \end{vmatrix}.$$

(Expansion along the first column.)

Application: Volume! See pages 748-9, Chapter 11 in the Calculus 250 text, or see Section 3.5 of your LA text.

Definition 3. If A is an $n \times n$ matrix, let A_{ij} be the $(n-1) \times (n-1)$ matrix formed by deleting the i^{th} row and j^{th} column of A. (This is different than your text's notation.) Then define

$$|A| = a_{11}|A_{11}| - a_{21}|A_{21}| + a_{31}|A_{31}| \cdots \pm a_{n1}|A_{n1}|$$

$$= \sum_{i=1}^{n} (-1)^{i+1} a_{i1} |A_{i1}|.$$

This is called "expansion along the first column."

Practice Problem 1. Find the determinants for the two matrices below by expanding along the first column.

$$A = \begin{bmatrix} 4 & -4 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 4 \\ 0 & -3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & 1 & 2 & 1 \\ 2 & -2 & 5 & 2 \\ 3 & 3 & 0 & 0 \end{bmatrix}$$

Solutions: $\det(A) = 75$, $\det(B) = -135$.

Remark 1. Determinants are used to define "volume" in higher dimensional spaces.

Theorem 1 (Important Properties of Determinants)**.** The following properties hold.

(0) $\det A^T = \det A$.
(1) If A' is obtained from A by switching two rows, then $\det A' = -\det A$.
(2) If A' is obtained from A by multiplying a row by a constant c, then $\det A' = c \det A$.
(3) If A' is obtained from A by adding the multiple of one to another then $\det A' = \det A$. (This is surprising!)

Proof. See appendix. [Not included.] □

Examples: We shall do several examples in class that will underscore the reasonableness of these properties.

Theorem 2 (Some Consequences)**.** The following consequences are easily derived from Theorem 1.

(1) If two rows are the same then $\det A = 0$ This follows from (1).
(2) Statements (1-3) remain true if "rows" is replace with "columns". Use (0).
(3) We can compute a determinant by expanding along any row or column, provided we watch our signs (see below). This follows from (0) and (1).

Example 1.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ h & i & j \end{vmatrix} = - \begin{vmatrix} d & e & f \\ a & b & c \\ h & i & j \end{vmatrix} = - \begin{vmatrix} d & a & g \\ e & b & h \\ f & c & i \end{vmatrix} =$$

$$- \left(d \begin{vmatrix} b & h \\ c & i \end{vmatrix} - e \begin{vmatrix} a & g \\ c & i \end{vmatrix} + f \begin{vmatrix} a & g \\ b & h \end{vmatrix} \right) =$$

$$-d \begin{vmatrix} b & h \\ c & i \end{vmatrix} + e \begin{vmatrix} a & g \\ c & i \end{vmatrix} - f \begin{vmatrix} a & g \\ b & h \end{vmatrix}.$$

But this is the same as expanding along the second row, but with the signs switched. In general the "checkerboard" patterns below tells us how the signs go.

n odd:

$$\begin{vmatrix} + & - & + & \cdots & - & + \\ - & + & - & \cdots & + & - \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ + & - & + & \cdots & - & + \end{vmatrix}$$

n even:

$$\begin{vmatrix} + & - & + & \cdots & + & - \\ - & + & - & \cdots & - & + \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ - & + & - & \cdots & + & - \end{vmatrix}$$

Practice Problem 2. Repeat the determinant calculations for the matrices A and B given in Practice Problem 1 by expanding along different rows and columns.

Example 2 (Upper triangular matrices)**.** Calculations of determinants can often be simplified by applying row and column operations. In the following example (from Len Evans', *A Brief Course in Linear Algebra*) identify the operation being used.

$$\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 3 & 0 & 1 & 1 \\ -1 & 6 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & -6 & 4 & -2 \\ 0 & 8 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & -5 & -5 \end{vmatrix} =$$

$$-5 \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 1 & 1 \end{vmatrix} = +5 \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 7 & 4 \end{vmatrix} = +5 \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{vmatrix}.$$

The last matrix is an upper triangular matrix. Its determinant is especially easy to compute.

$$\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{vmatrix} = 1 \cdot 2 \cdot \begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} = 1 \cdot 2 \cdot 1 \cdot (-3) = -6.$$

Thus the determinant of the original matrix is $5 \cdot (-6) = -30$.

Fact 1. For any upper triangular matrix the determinant is just the product of the diagonal elements. The determinant of any matrix can be found by first putting it into upper triangle form. Computer programs using this approach are faster than ones using row or column expansion.

2. Two Big Theorems

Theorem 3. A^{-1} exists if and only if $\det A \neq 0$.

Theorem 4. $\det AB = \det A \det B$.

Remark 2. Both of these can be made intuitively plausible by thinking in terms of area (or volume).

Proof of Theorem 3. The square matrix A is nonsingular (i.e. invertible) if and only if there exists a sequence of row operations taking A to I.

If $\det A = 0$ any matrix derived from A by row operations will also have zero determinant. Hence A is not row equivalent to I and so A^{-1} does not exist.

Suppose now that A is known to be not invertible. Let B be the reduced matrix derived from A. Then B cannot have a complete set of pivots, that is B must have a zero on its diagonal. But B is an upper triangular matrix (because it is reduced). Thus $\det B = 0$, which implies $\det A = 0$. $\square$

Proof of Theorem 4. First we establish the following fact. Let A and B be matrices such that B can be derived from A by a single row operation which we denote by r, i.e.

$$A \xrightarrow{r} B.$$

Now let C be a third matrix and consider the products AC and BC. Our claim is that if you apply the same row operation r to AC you get BC,

$$AC \xrightarrow{r} BC.$$

I will show this for row operation 3 for 3×3 matrices. You should be able to see how this could be extended to cover all the other cases.

Let

$$A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}, \qquad B = \begin{bmatrix} R_1 \\ R_2 \\ R_3 + kR_2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}.$$

Then

$$AC = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix}$$

and

$$BC = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ (R_3+kR_2)C_1 & (R_3+kR_2)C_2 & (R_3+kR_2)C_3 \end{bmatrix} =$$

$$\begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1+kR_2C_1 & R_3C_2+kR_2C_2 & R_3C_3+kR_2C_3 \end{bmatrix}.$$

Thus original row operation takes AC to BC. We are now in position to prove Theorem 2. The prove is divided into two cases.

Case 1: Suppose A is nonsingular ($\det A \neq 0$). Then there are k row operations taking A to I, for some number k:

$$A = A_0 \xrightarrow{r_1} A_1 \xrightarrow{r_2} A_2 \xrightarrow{r_3} \cdots \xrightarrow{r_k} A_k = I$$

For each r_i there is a nonzero number c_i (which could be 1) such that $\det A_{i-1} = c_i \det A_i$. Thus,

$$\det A = c_1 \det A_1 = c_1c_2 \det A_2 = \cdots = c_1c_2c_3\cdots c_k \det I,$$

and so we can write $\det A = c_1c_2\cdots c_k$. Now apply exactly the same row operations to the product AB.

$$AB \xrightarrow{r_1} A_1B \xrightarrow{r_2} A_2B \xrightarrow{r_3} \cdots \xrightarrow{r_k} IB.$$

Thus we have

$$\det AB = c_1\cdots c_k \det B = \det A \det B.$$

Case 2: Suppose $\det A = 0$. We must show that $\det AB = 0$ since $\det A \det B = 0 \cdot \det B = 0$.

Since A is not invertible there is a sequence of row operations taking A to a matrix Z that has a row of zeros. (Why? A good test question!?)

$$A = A_0 \xrightarrow{r_1} A_1 \xrightarrow{r_2} A_2 \xrightarrow{r_3} \cdots \xrightarrow{r_k} A_k = Z.$$

Thus,

$$AB \xrightarrow{r_1} A_1B \xrightarrow{r_2} A_2B \xrightarrow{r_3} \cdots \xrightarrow{r_k} ZB.$$

Then

$$\det AB = c_1\cdots c_k \det ZB.$$

But if Z has a row of zeros so does ZB (check this!). Thus, $\det ZB = 0$. This completes our proof. □

3. Inverses and Cramer's Rule

Cramer's Rule is another method for finding the inverse of a matrix. For larger systems it is an inefficient method. Row reduction to an upper triangular matrix is best. However, if the matrix entries are variables or functions row reduction may fail since you don't know if an entry is zero or nonzero.

Definition 4. Given an $n \times n$ matrix A the ij **cofactor** is given by

$$c_{ij} = (-1)^{i+j}\det(A_{ij}),$$

where we recall that A_{ij} was obtained from A by deleting row i and column j.

Definition 5. The **adjoint matrix** of A is the matrix formed from its cofactors.

$$\operatorname{adj}(A) = [c_{ij}]^T$$

Theorem 5. If $\det(A) \neq 0$ then

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}$$

Proof. We shall skip the proof. □

Practice Problem 3. Use Theorem 5 to find the inverses of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{and } B = \begin{bmatrix} x & 2\,x & 3 \\ x^2 & 3 & 6\,x \\ 2-x & x & 0 \end{bmatrix}$$

Solution:

$$A^{-1} = \begin{bmatrix} \dfrac{d}{a\,d-b\,c} & \dfrac{-b}{a\,d-b\,c} \\ \dfrac{-c}{a\,d-b\,c} & \dfrac{a}{a\,d-b\,c} \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} \dfrac{2x^2}{D} & \dfrac{-x}{D} & \dfrac{-4x^2-3}{D} \\ \dfrac{2x(x-2)}{D} & \dfrac{2-x}{D} & \dfrac{x^2}{D} \\ \dfrac{x^3+3x-6}{-3D} & \dfrac{x(3x-4)}{3D} & \dfrac{x(2x^2-3)}{3D} \end{bmatrix}$$

where $D = 5\,x^3 - 8\,x^2 + 6 - 3\,x$.

This method of finding inversing can be modified to find solutions to $n \times n$ systems of equations, provided there is a unique solution. This method is called **Cramer's Rule**, and is commonly employed in Engineering courses. See Theorem 2.13 and Example 8 in your textbook on pages 112-3.

4. Alternative Definition

There is another method for finding determinants. Before we consider it we must briefly discuss permutations. A **permutation** is a function that rearranges the elements of a list of numbers. Thus, $p : (1,2,3,4,5) \longrightarrow (3,2,1,5,4)$ is a permutation. The set of all possible permutations of the list $(1,2,3,\ldots,n)$ will be called P_n.

Homework Problem 4. List the elements of P_n for n equal to 2, 3 and 4.

Homework Problem 5. Prove that P_n has $n!$ members. Hint: use induction.

The simplest permutation is one that just switches two entries in a list. Every permutation can be broken down into a sequence of switches. For example

$$(1,2,3,4,5) \longrightarrow (3,2,1,4,5) \longrightarrow (3,2,1,5,4)$$

and

$$(1,2,3,4,5) \longrightarrow (2,1,3,4,5) \longrightarrow (2,3,1,4,5) \longrightarrow (2,3,1,5,4) \longrightarrow (3,2,1,5,4)$$

are two ways to break p down into switches.

Practice Problem 4. Problem: Consider $q : (1, 2, 3, 4, 5, 6) \longrightarrow (3, 4, 2, 1, 5, 6)$. Break q down into switches several different ways.

For any given permutation the number of switches used to create it is always even or always odd. We shall not prove this. If you take math 319 you will do the proof there. The **parity** of a permutation is 0 if it decomposes into an even number of switches and 1 if it decomposes into an odd number of switches. The parity function is denoted by σ. Thus, $\sigma(p) = 0$ and $\sigma(q) = 1$.

We will be applying permutations to entries of matrices. For example if $A = [a_{ij}]^{6\times 6}$. then $a_{q(2)4} = a_{44}$ and $a_{q(1)q(6)} = a_{36}$.

We are now ready to present the alternative definition of determinants.

Definition 6. The determinant of an $n \times n$ matrix is given by

$$\det(A) = \sum_{p \in P_n} (-1)^{\sigma(p)} a_{1p(1)} a_{2p(2)} a_{3p(3)} \cdots a_{np(n)}$$

Check that for 3×3 matrices this becomes

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

Theorem 6. Definitions 3 and 6 of the determinant are equivalent.

Remark 3. We shall not do the proof, but it is easy to check in the 2×2 and 3×3 cases. The proof of the general case is difficult.

Practice Problem 5. Find the determinants of A and B given in Practice Problem 1 using this alternative definition of a determinant.