Lines and Planes[1]

1 Lines in the Plane

Every line of points L in $\mathbb{R}^2$ can be expressed as the solution set for an equation of the form $Ax + By = C$. The equation is not unique for if we multiply both sides by any nonzero number the solution set is unchanged. Any line L can also be expressed by a pair of parametric equations of the form:

$$\begin{aligned} x(t) &= at + b \\ y(t) &= ct + d \end{aligned}$$

These can be rewritten in vector form: $\langle x, y\rangle = \langle a, c\rangle\, t + \langle b, d\rangle$. The vectors $\langle a, c\rangle$ and $\langle b, d\rangle$ have a nice geometric/physical interpretation. Regard t as time. Let $\mathbf{p}(t) = \langle x(t), y(t)\rangle$ and call it the *position vector*. One can imagine a particle moving along L in accordance with the given parametric equations. Then $\mathbf{p}(0) = \langle b, d\rangle$ is the *initial position*. Notice,

$$\frac{d\mathbf{p}}{dt} = \langle x'(t), y'(t)\rangle = \langle a, c\rangle$$

Thus, we call $\mathbf{v} = \langle a, c\rangle$ the *velocity vector*. It is parallel to L. It is customary to place its base point on L. See Figure 1(left side).

We now give a geometric interpretation for the "ABC" form of an equation of a line. First, suppose $C = 0$; this just means the line L goes through the origin. Let $\mathbf{n} = \langle A, B\rangle$, and again set $\mathbf{p} = \langle x, y\rangle$. Then we have $\mathbf{n} \bullet \mathbf{p} = 0$. That is the vectors $\mathbf{n}$ and $\mathbf{p}$ are at right angles to each other. Thus, the line L for $Ax+By = 0$ is the set of all points (x, y) such that $\langle x, y\rangle$ is perpendicular to $\langle A, B\rangle$.

Now we consider the general case: $Ax + By = C$. Pick some particular point on the line and call it (x_0, y_0). Then $C = Ax_0 + By_0$. Therefore, for

---

any point (x, y) on L we have $Ax + By = Ax_0 + By_0$. We can rewrite this as

$$\begin{aligned} Ax - Ax_0 + By - By_0 &= 0 \\ A(x - x_0) + B(y - y_0) &= 0 \\ \langle A, B\rangle \bullet \langle x - x_0, y - y_0\rangle &= 0 \\ \mathbf{n} \bullet (\langle x, y\rangle - \langle x_0, y_0\rangle) &= 0 \\ \mathbf{n} \bullet (\mathbf{p} - \mathbf{p}_0) &= 0 \end{aligned}$$

In the last line we have let $\mathbf{p}_0 = \langle x_0, y_0\rangle$. The vector $\mathbf{p} - \mathbf{p}_0$ can be thought of as lying in L with its tail at (x_0, y_0) and its head at (x, y).

Thus, L is the unique line perpendicular to the vector $\mathbf{n} = \langle A, B\rangle$ that passes through (x_0, y_0). See Figure 1(right side). The vector $\mathbf{n}$ is called a *normal vector* for the line L. Given a vector to use as normal vector and a point we can easily find an equation for the corresponding line.

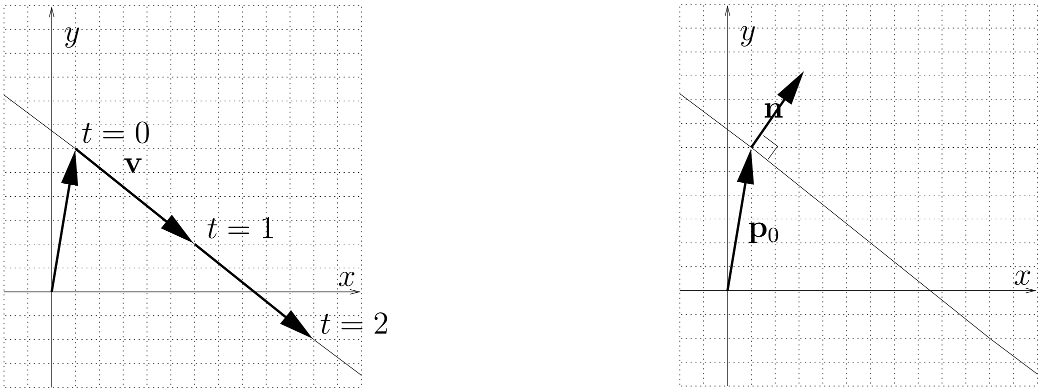

Figure 1: Left: A parametric line. Right: Normal vector to a line.

Problem 1. Consider the line determined by $x(t) = 3t - 2$ and $y(t) = -t + 7$. Find an equation for the line in ABC form.

Problem 2. Consider the line determined by $4x - 7y = 2$. Find a pair of parametric equations for this line.

Note: Both of these problems have many correct answers.

2 Lines and Planes in 3-space

The three dimensional set $\mathbb{R}^3$ is the set of all triples (x, y, z) where x, y, and z are real numbers. Such a triple is called the xyz-coordinates of a point. These are also called *rectilinear coordinates*. The set $\{(x, 0, 0) \,|\, x \in \mathbb{R}\}$ is the x-axis. The y and z axes are defined similarly. They are clearly lines. The set $\{(x, y, 0) \,|\, x \in \mathbb{R}\,,\, y \in \mathbb{R}\}$ is the xy-plane. The yz and xz planes are defined similarly. Visualizing structures in three dimensions takes practice.

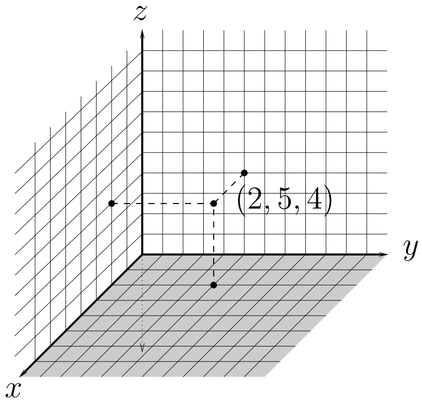

Figure 2: Three dimensional space: $\mathbb{R}^3$

Any line L in $\mathbb{R}^3$ can expressed parametrically in the form:

$$\begin{aligned} x(t) &= at + b \\ y(t) &= ct + d \\ z(t) &= et + f \end{aligned}$$

or, in vector form, $\langle x, y, z\rangle = \langle a, c, e\rangle\, t + \langle b, d, f\rangle$. As with lines in $\mathbb{R}^2$ it is useful to think of $\langle a, c, e\rangle$ as a velocity vector and $\langle b, d, f\rangle$ as the position at $t = 0$.

However, there is no way to express a line in $\mathbb{R}^3$ as a single equation in three variables. In fact, we will show that “typically” solution sets of equations of the form $Ax + By + Cz = D$ are planes and that every plane in $\mathbb{R}^3$ is the solution set of some equation in this form. Note: If $A = B = C = D = 0$, the solution set is all of $\mathbb{R}^3$; if $A = B = C = 0$ but $D \neq 0$ the solution set is empty.

Example 1. Convince yourself of the following:

- If $A = B = D = 0$ and $C \neq 0$ then $Ax + By + Cz = D$ is the xy-plane.
- If $A = C = D = 0$ and $B \neq 0$ then $Ax + By + Cz = D$ is the xz-plane.
- If $B = C = D = 0$ and $A \neq 0$ then $Ax + By + Cz = D$ is the yz-plane.

Let's consider the case where $D = 0$. Let $\mathbf{n} = \langle A, B, C \rangle$ and $\mathbf{p} = \langle x, y, z \rangle$. Then the equation $Ax + By + Cz = 0$ becomes $\mathbf{n} \bullet \mathbf{p} = 0$. Thus, the solution set is the plane P, passing through the origin of $\mathbb{R}^3$ whose points, when regarded as vectors, are perpendicular to $\mathbf{n}$.

We return to the general case: $Ax + By + Cz = D$. Let $\mathbf{p}_0 = \langle x_0, y_0, z_0 \rangle$ be some fixed point that satisfies the given equation. We leave it to the reader to show that

$$\mathbf{n} \bullet (\mathbf{p} - \mathbf{p}_0) = 0.$$

Thus, the solution set of $Ax + By + Cz = D$ is the unique plane passing through $\mathbf{p}_0$ and perpendicular to $\mathbf{n} = \langle A, B, C \rangle$.

Example 2. Consider $x + y + z = 1$. The points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ satisfy the equation. We can connect them with line segments and visualize the triangle thus formed. This triangle sits in the plane. If we place the tail of $\mathbf{n} = \langle 1, 1, 1 \rangle$ at any point of the triangle it is easy to see that it is perpendicular to the plane.

Example 3. Let P_1 be the plane given by $2x + 3y - z = 5$ and let P_2 be the plane given by $x + y + z = 1$. Find parametric equations for the line $L = P_1 \cap P_2$.

Solution.

$$\left.\begin{aligned} 2x + 3y - z &= 4 \\ x + y + z &= 1 \end{aligned}\right\} \implies y - 3z = 2.$$

Let $z = t$. Then $y = 3t + 2$. Also, $x = 1 - y - z = -4t - 1$. We rewrite these as $\langle x, y, z \rangle = \langle -4, 3, 1 \rangle\, t + \langle -1, 2, 0 \rangle$. $\square$

Example 4. Find an equation for the plane passing through the three points $(1, 1, 1)$, $(1, 2, 3)$, and $(2, -1, 0)$.

Solution. We have three conditions and these give us three equations in four unknowns.

$$\left.\begin{array}{rcrcrcr} A & + & B & + & C & = & D \\ A & + & 2B & + & 3C & = & D \\ 2A & - & B & & & = & D \end{array}\right\} \Longrightarrow$$

$$\left.\begin{array}{rcrcrcr} A & + & B & + & C & = & D \\ & & B & + & 2C & = & 0 \\ & & -3B & - & 2C & = & -D \end{array}\right\} \Longrightarrow$$

$$\left.\begin{array}{rcrcrcr} A & + & B & + & C & = & D \\ & & B & + & 2C & = & 0 \\ & & & & C & = & -D/4 \end{array}\right\} \Longrightarrow$$

$$\left.\begin{array}{rcrcrcr} A & + & B & & & = & 5D/4 \\ & & B & & & = & D/2 \\ & & & & C & = & -D/4 \end{array}\right\} \Longrightarrow \begin{array}{rcr} A & = & 3D/4 \\ B & = & D/2 \\ C & = & -D/4 \end{array}$$

Any nonzero value of D will do. Let $D = 4$. Then $3x + 2y - z = 4$ is an equation for our plane. □

Problem 1. Consider the three points $(1, 1, 1)$, $(2, 0, 2)$, and $(4, -1, 4)$. Show that they do not determine a unique plane because they lie on the same line. Find an equation for this line; write it in vector form.

Problem 2. Let P be the plane given by $x + 2y - 3z = 1$. Let L_{xy} be the intersection of P with the xy-plane. Define L_{xz} and L_{yz} similarly. Find equations for these three lines in "ABC" form.

Problem 3. Graph, separately, each of the planes determined by these three equations: $2x + 2y - 3z = 1$, $x + 2y + 4z = -1$, and $3x - 2y - 2z = 7$.

Problem 4. Find the point of intersection of the three planes determined by these three equations: $2x+2y-3z = 1$, $x+2y+4z = -1$, and $3x-2y-2z = 7$.

Problem 5. Show that the two planes determined by $2x + 2y - 3z = 1$ and $4x + 4y - 6z = 0$ do not intersect and are thus parallel.

Problem 6. Let P be the plane given by $2x + 3y - 2z = 1$. Let L be the line given by $\langle x, y, z\rangle = \langle 1, 1, 1\rangle\, t + \langle 1, 0, 1\rangle$. Find the point where they meet.

Problem 7. Show that these four points lie in the same plane: $(1, 1, -1)$, $(-1, 0, 0)$, $(-1, 1, -\frac{1}{2})$, and $(1, -1, 0)$. Find an equation for this plane.

3 Parametric Equation for a Plane

There is another form for equations of planes in $\mathbb{R}^3$ that is the analog of the parametric form for equations of a line. The difference is we will need two parameters, r and s, instead of one. Of course, the time metaphor is no longer useful.

Let P be a plane given by $Ax+By+Cz=D$. Assume that $C \neq 0$. Then we can solve for z and get $z = D/C - A/Cx - B/Cy$. (If $C = 0$ solve for x or y instead.) Think of z as the height above the xy-plane. Now let $x = r$ and $y = s$, and think of r and s as free parameters. We can now write

$$\begin{aligned} \langle x, y, z\rangle &= \langle r, s, D/C - A/Cr - B/Cs\rangle \\ &= \langle 0, 0, D/C\rangle + r\,\langle 1, 0, -A/C\rangle + s\,\langle 0, 1, -B/C\rangle \end{aligned}$$

This equation is far from unique. We can start with any point $(x_0, y_0, z_0) \in P$, regard it as a vector $\mathbf{p}_0 = \langle x_0, y_0, z_0\rangle$ and add multiplies of $\langle 1, 0, -A/C\rangle$ and $\langle 0, 1, -B/C\rangle$ to it and stay in the plane. Furthermore, if we let $\mathbf{v}_1$ and $\mathbf{v}_2$ be nonzero multiplies of $\langle 1, 0, -A/C\rangle$ and $\langle 0, 1, -B/C\rangle$, respectively then

$$\mathbf{p} = \mathbf{p}_0 + r\mathbf{v}_1 + s\mathbf{v}_2$$

gives the same plane P. Indeed, we could use any pair of vectors in P with tails at $\mathbf{p}_0$ as long as they point in different directions.

We will use this formulation to place a coordinate system on P. Take a point (x_0, y_0, z_0) on P and call it the origin of P. Then any point on P can be gotten to by adding multiplies of $\mathbf{v}_1$ and $\mathbf{v}_2$ to $\mathbf{p}_0$. Thus, for any point on P we can think of it as having coordinates (r, s). See Figure 3.

Example 1. Define a plane P by

$$\langle x, y, z\rangle = \langle 1, 2, 3\rangle + r\,\langle 1, 1, 0\rangle + s\,\langle 0, 1, 1\rangle$$

Show that the point (0,2,4) is on P and find its rs-coordinates.

Solution. We have three equations and two unknowns.

$$\left.\begin{aligned} 0 &= 1 + 1r + 0s \\ 2 &= 2 + 1r + 1s \\ 4 &= 3 + 0r + 1s \end{aligned}\right\} \implies \begin{aligned} r &= -1 \\ s &= 1 \end{aligned}$$

Thus, $(0, 2, 4) \in P$ and it has rs-coordinates $(-1, 1)$ relative to the given parametric equation. □

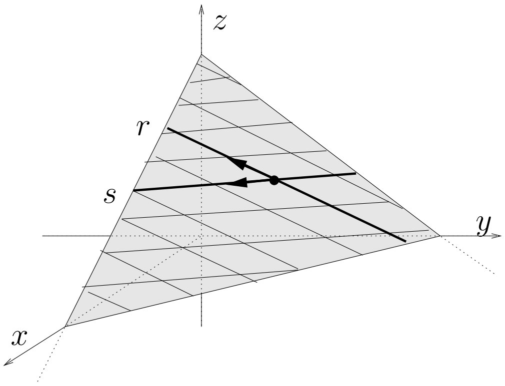

Figure 3: Coordinates for a plane: The dark lines are the r and s-axes

Problem 1. Using the same plane P in Example 1, find the rs-coordinates of $(3, 3, 2)$.

Problem 2. Show that the point $(1, 2, -1)$ is not on the plane P of Example 1.

Problem 3 (Hard). The equation $2r + 3s = 1$ determines a line L in the plane P of Example 1, using rs-coordinates. Find a parametric equation for L in xyz-coordinates.