2×2 Matrix Theory[1]

In this section we develop some the basic concepts of matrix theory in the special case of 2×2 matrices. In this section all matrices are 2×2 unless stated otherwise. Readers with a fairly strong mathematics background can probably skip this section. Instructors may wish to assign it as outside reading and not spend class time on it, or may decide to delay covering it until Section **??**.

Matrix addition and scalar multiplication of matrices behave just like vector addition and scalar multiplication of vectors. For any size $m \times n$ the matrix of all zeros is the identity element for matrix addition. Matrix multiplication is another story. We have already seen cases where $AB \neq BA$. But it gets worse.

We give several examples and problems that illustrate the differences between matrix algebra and the ordinary algebra of real numbers. Then we show how with a deeper understanding of matrix structure the two algebras are analogous after all.

Example 1. For real numbers $ab = 0$ implies $a = 0$ or $b = 0$. Let $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then O is the identity element for addition of 2×2 matrices. It is called the (2×2) **zero matrix**. But, $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} = O$.

A matrix is **nonzero** if it is not O, that is if it has at least one nonzero entry. There are nonzero matrices A such that $A^2 = O$.

Problem 1. Find a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq O$ such that $A^2 = O$.

Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. It is easy to see that $AI = A$ and $IA = A$. Thus, I serves as an identity element for 2×2 matrix multiplication.

Example 2. For real numbers $a^2 = 1$ implies $a = \pm 1$. But, $\begin{bmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Example 3. For real numbers $a^2 = a$ implies $a = 1$ or $a = 0$. But $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Next we consider problems of the form $A\mathbf{x} = \mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are column vectors. In longhand this is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

which is equivalent to the system of equations

$$\begin{aligned} ax_1 + bx_2 &= y_1, \\ cx_1 + dx_2 &= y_2. \end{aligned}$$

We assume A is given. If $\mathbf{y}$ is also given, the goal is to solve for $\mathbf{x}$. If $\mathbf{x}$ and $\mathbf{y}$ are both regarded as variable vectors, we can think of A as defining a function, $A : \mathbb{R}^2 \to \mathbb{R}^2$. Both points of view are interesting and useful.

Example 4. For real numbers $a \neq 0$ we have that $ax = ay$ implies $x = y$. But $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

For real numbers the problem $ax = b$ has a unique solution for x if $a \neq 0$. But for $a = 0$ and $b = 0$ we get $0x = 0$, which has infinitely many solutions, or if $b \neq 0$, then $0x = b$ has no solutions.

Problem 2. Construct two examples each such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$ has (a) a unique solution, (b) no solutions, and (c) infinitely many solutions. (Here x and y are variables; you are to chose the constants a, b, c, d, e, and f. This problem is tedious but important.)

The *determinant* of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be $ad - bc$. The determinant of (any square matrix) A is denoted by $\det A$ or by $|A|$.

Problem 3. Compute the determinant for each of the 2×2 matrices you used in Problem 2. What do you notice?

Problem 4. Prove the following statements for 2×2 matrices. Assume $\mathbf{b}$ and $\mathbf{v}$ are column vectors.
(a) If $\det A \neq 0$ then $A\mathbf{v} = \mathbf{b}$ has a unique solution.
(b) If $\det A = 0$ then $A\mathbf{v} = \mathbf{b}$ has either no solutions or infinitely many solutions.

Problem 5. Prove that for 2×2 matrices $\det AB = \det A \det B$.

Problem 5 has some interesting consequences. Suppose $\det A \neq 0$. Then $\det A^2 \neq 0$. Since $\det O = 0$ we can only have $A^2 = O$ if $\det A = 0$. It turns out that much of the seemingly pathological behavior in 2×2 matrix algebra can only occur when the determinant is zero. An exception is the lack of commutativity for matrix multiplication; this can occur even when both matrices have nonzero determinants. The matrices in Example 2 have nonzero determinants. But by Problem 5 if $A^2 = I$ then $\det A = \pm 1$.

A 2×2 matrix A is **invertible** if there is another matrix B such that $AB = I$. In this case B is called the **inverse** of A and is denoted A^{-1}. In a widely used alternative terminology invertible matrices are called **nonsingular**, and noninvertible matrices are called **singular**.

Problem 6. It can be shown that for square matrices if $AB = I$ then it follows that $BA = I$, even though matrix multiplication does not commute in general. Prove this claim for 2×2 matrices.

Problem 7. Prove that the inverse of a 2×2 matrix A exists if and only if $\det A \neq 0$. Find a formula for A^{-1} when it exists. (Remember this formula!)

Problem 8. Let A be an invertible 2×2 matrix. Prove that $\det A^{-1} = \dfrac{1}{\det A}$.

Problem 9. For A an invertible matrix, prove that A^{-1} is invertible and find its inverse.

Problem 10. Prove that $\det A = \det A^T$.

Problem 11. Suppose A and B are invertible and 2×2. Prove that AB is invertible and that $(AB)^{-1} = B^{-1}A^{-1}$.

Problem 12. Assuming A is invertible and 2×2, prove that $A\mathbf{x} = A\mathbf{y}$ implies $\mathbf{x} = \mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are column vectors.

Remark. The conclusions to Problems 7 to 12 are valid for $n \times n$ matrices as we will see in Chapter **??**.

Inverses, when they exist, give us a new way to solve systems of equations. If we want to solve $A\mathbf{v} = \mathbf{b}$ and A^{-1} exists we can multiply through by A^{-1}.

$$\begin{aligned} A\mathbf{v} &= \mathbf{b} && (1)\\ A^{-1}A\mathbf{v} &= A^{-1}\mathbf{b} && (2)\\ I\mathbf{v} &= A^{-1}\mathbf{b}\\ \mathbf{v} &= A^{-1}\mathbf{b} \end{aligned}$$

There are two unstated assumptions in the above derivation. It is easy to check that $I\mathbf{v} = \mathbf{v}$. But is it really clear that (1) $\Longleftrightarrow$ (2)? Yes, this follows from Problems 9 and 12. (You have done these, right?)

Problem 13. Solve $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ using inverses.

But why should the determinant play this role? What does the determinant really *mean*? To find out we recall the idea that $A\mathbf{x} = \mathbf{y}$ can be thought of as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. If it is one-to-one and onto then it is invertible. For functions from $\mathbb{R}$ to $\mathbb{R}$ you have graphical ways of seeing one-to-oneness (the horizontal line test). For functions of the form $y = ax$ it is clear that they are one-to-one and onto if and only if $a \neq 0$. When $a = 0$ the range is only one point, $\{0\}$.

Let $S = [0,1] \times [0,1]$ denote the unit square in $\mathbb{R}^2$. It has corners: (0,0), (1,0), (1,1), and (0,1). Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We will study the image of S under the function $A\mathbf{x} = \mathbf{y}$. Denote the image by $A(S)$. Let's work through some examples.

Example 5. Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. The effect of this matrix on the unit square is shown in Figure 1. You should be able to verify that $A(S) = [0,3] \times [0,2]$, a rectangle. What happens if we replace the 3 with a -3, or the 2 with $1/2$? Experiment with various diagonal matrices. In each case notice that the area of $A(S)$ is $|\det(A)|$, the absolute value of $\det A$.

Problem 14. Let $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$. Prove that $A : \mathbb{R}^2 \to \mathbb{R}^2$ is one-to-one and onto if and only if $\det A \neq 0$.

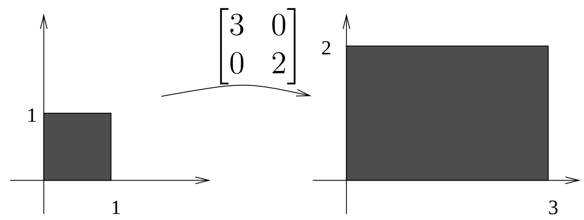

Figure 1: Area of $A(S)$ is six.

Example 6. Consider matrices of the form $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, where a and b are both nonzero. The image $A(S)$ is no longer solid region but is rather a line segment in the x-axis from $(0, 0)$ to $(a + b, 0)$. Figure 2 gives an example. As a function A is not onto. For example,

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

has no solutions, as you can check. It is not one-to-one since, for example,

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{a}{b} \end{bmatrix}.$$

Problem 15. Let $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$. Prove that each point $(x, 0)$ in $A(S)$ has infinitely many *preimages*, *i.e.* points in $\mathbb{R}^2$ that get mapped to it.

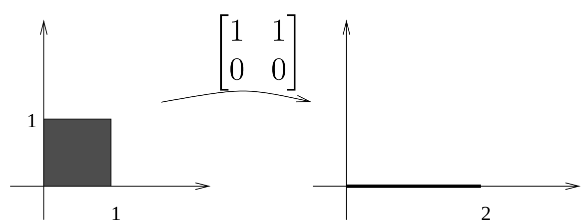

Figure 2: $A(S)$ is the line segment from (0,0) to (2,0).

Problem 16. Study the images of matrices of the form $\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$. Figure 3 gives an example.

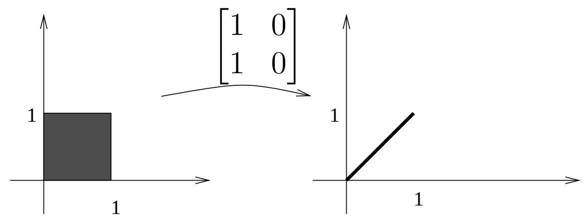

Figure 3: $A(S)$ is the line segment from (0,0) to (1,1).

Example 7. Figure 4 shows $A(S)$ for $A = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}$. The image of S is a parallelogram.

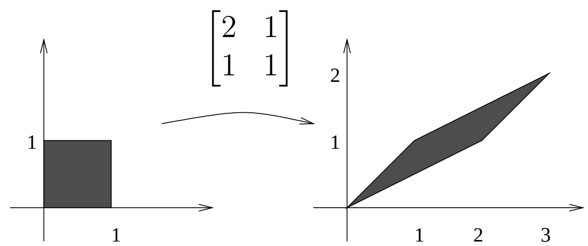

Figure 4: $A(S)$ is a parallelogram.

Problem 17. Let A be a 2×2 matrix with nonzero determinant and S be the unit square in $\mathbb{R}^2$.
(a) Let L be a line in $\mathbb{R}^2$. Prove that $A(L)$ is a line.
(b) Prove that $A(S)$ is a parallelogram.
(c) Prove that $A(S)$ has area $|\det(A)|$.
(d) Let R be any rectangle in the plane with sides parallel to the coordinate axes. Prove that $A(R)$ is a parallelogram with area equal to the area of R multiplied by $|\det(A)|$.

Problem 18. Let A be a 2×2 matrix with $\det A = 0$. Prove that $A(\mathbb{R}^2)$ is either $\{(0,0)\}$ or is a line. Prove that each point in $A(\mathbb{R}^2)$ has infinitely many preimages.

Remark (Philosophical Conclusion)**.** We can now see why the determinant plays such an important role in matrix algebra. If the determinant of A is zero, then the matrix, as a function, takes the 2-dimensional plane to a set of smaller "dimension," and, not surprisingly, the function is not one-to-one or onto, and thus is not invertible. Zero determinant matrices play a role in 2×2 matrix algebra much like the role zero plays in ordinary algebra. If the determinant of A is not zero, the function A is one-to-one and onto, and thus is invertible. It turns out that these ideas carry over directly to the study of $n \times n$ matrices. We leave you with one last thought. The determinant measures how a unit of area is stretched and/or contracted by a matrix function. This is kind of like a rate of change. Multiplying two matrices is equivalent to composing the functions they give. In this way the rule $\det AB = \det A \det B$ (recall Problem 5) is parallel to the chain rule, $D(f \circ g) = Df \cdot Dg$, from differential calculus.

Problem 19. For each matrix draw $A(S)$. (a) $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 4 \\ -2 & -8 \end{bmatrix}$

Problem 20. Let $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Prove that R_θ rotates vectors about the origin through angle θ. What is $\det R_\theta$? (Matrices of this from are called **rotation matrices**.