Bases and Dimension [1]

For each n the vector $\mathbb{R}^n$ has a nice structure. Every point can be thought of a linear combination the elementary vectors: $\mathbf{e}_1 = (1, 0, 0, ..., 0)$, $\mathbf{e}_2 = (0, 1, 0, ..., 0)$,...,$\mathbf{e}_n = (0, 0, ..., 0, 1)$. The set $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ is called a *basis* for $\mathbb{R}^n$. The goal of this section is to describe an analogous structure for a general vector space. This will enable us to make a precise definition of the dimension of a vector space which will it turn allow to show that the number of free variables used to parametrize the solution set of a system of linear equations is independent of how the solution set was found.

Definition .1. Let B be a subset of a vector space V. If the span of B is V and B is linearly independent, then we say B is a **basis** for V. If B is finite, $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then V is **finite dimensional** with **dimension** equal to n, the number of vectors in the basis B. If B is infinite, then V is **infinite dimensional**.

Every vector space has a basis. However, the proof of this fact uses some subtle concepts and is much too difficult to do here. (See: *Algebra*, by Serge Lang, 3rd Ed., page 139, Theorem 5.1.) We shall only be concerned with finite bases, although some interesting vector spaces have infinite bases: the space of all polynomials has $\{1, x, x^2, x^3, \ldots\}$ as a basis. A difficult question is, what would be a basis for the set of all functions from $\mathbb{R}$ to $\mathbb{R}$?

Remark. We regard a basis B as an ordered set. Thus, $\{e_1, e_2\}$ and $\{e_2, e_1\}$ are different bases for the plane $\mathbb{R}^2$.

For the definition of dimension to make sense we must show that given any two bases for a vector space the number of vectors in each basis is the same. Otherwise, a space could have two different dimensions! First we give some examples.

Example 1. The set $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is not a basis for $\mathbb{R}^2$ since it does not span $\mathbb{R}^2$, but it is a basis for the subspace determined by the equation $y = x$. The set $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$. The set $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ is not a basis since it is linearly dependent.

Example 2. The solution set S to $y''+y=0$ is spanned by $B=\{\sin x, \cos x\}$. Since, B is linearly independent, $W(\sin x, \cos x)=1$, we have that B is a basis for S. Note: we have not proven that $S=\operatorname{span} B$; we have only checked that $\operatorname{span} B \subset S$; proof of their equality is a standard topic in differential equations courses.

Theorem .2. *If a vector space V has a finite basis $B \subset V$ with n members, then every other basis of V has n members. Thus, the concept of dimension is well defined.*

First we will prove Theorem .2 for the case $n=4$. Most students should be able to follow it. Then, for the adventuresome, we give the general proof. Statement 5 of Theorem **??** (page **??**) will be the key to both proofs.

Proof. $n=4$

□

Proof. For all n Let V be a vector space. Let $B_1=\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ be a basis for V with n members. Let $B_2=\{\mathbf{w}_1,\ldots,\mathbf{w}_m\}$ be another basis. Suppose $m>n$. We shall derive a contradiction.

Remark. The contradiction we shall derive only makes use of the assumption that B_2 is linearly independent. Thus, if B_2 were infinite we could derive the same contradiction by choosing a subset of B_2 with $m>n$ members and working only with it.

Every $\mathbf{w}_i \in B_2$ is a linear combination of vectors in B_1. For each $i=1,...,m$ write

$$\mathbf{w}_i=\sum_{j=1}^{n} c_{ij}\mathbf{v}_j$$

Let

$$C=\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ c_{m1} & \cdots & c_{mn} \end{bmatrix}=\begin{bmatrix} R_1 \\ \vdots \\ \vdots \\ R_m \end{bmatrix},$$

where each R_i is just defined to be the i-th row of C.

Let $D=\operatorname{rref}(C)$. The number of pivots is less than or equal to m the number of rows. Since C has more rows than columns $(m>n)$ D will have at least one row without pivots, that is D will have a row of zeros. This means that a sequence of row operations produce the zero row vector. Thus, there is a nontrivial linear combination of the rows of C is equal to the zero vector.

In other words, there are real number a_i, $i = 1, ..., m$, not all zero, such that

$$\sum_{i=1}^{m} a_i R_i = \mathbf{0} \qquad \text{(the zero vector).}$$

This means that for each $j = 1, ..., n$ (that is for each column of C) we have

$$\sum_{i=1}^{m} a_i c_{ij} = 0 \qquad \text{(the number zero).}$$

But, now consider $\sum\limits_{i=1}^{m} a_i \mathbf{w}_i$. We compute

$$\begin{aligned} \sum_{i=1}^{m} a_i \mathbf{w}_i &= \sum_{i=1}^{m} a_i \sum_{j=1}^{n} c_{ij} \mathbf{v}_j \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} a_i c_{ij} \mathbf{v}_j \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} a_i c_{ij} \mathbf{v}_j \\ &= \sum_{j=1}^{n} \mathbf{v}_j \sum_{i=1}^{m} a_i c_{ij} \\ &= \sum_{j=1}^{n} \mathbf{v}_j 0 = \mathbf{0} \end{aligned}$$

Thus, if $m > n$ the vectors in B_2 are not linearly independent, contradicting the given that B_2 is a basis for V. A similar contradiction is obtained if $m < n$. Therefore $m = n$. $\square$

TOO HARD:

Problem 1. Let A be a $m \times n$ matrix with $m > n$. Prove that $\mathrm{rref}\,(A)$ has at least one row of all zeros. Do so as follows. Explain why it is enough to do the case $m = n + 1$. Prove that the claim holds for $n = 1$. Then apply the principal of mathematical induction.

0.1 Facts about Bases

Here we state four facts about bases of a vector space. We shall prove two of them, leaving the other two as homework problems. Several examples follow. You may wish to study the examples before tackling the proofs. Let V be a finite dimensional vector space with $S \subset V$ a finite subset that does not contain the zero vector.

Proposition .3. *If* $\operatorname{span} S = V$*, but* S *is not linearly independent, then some subset* $S' \subset S$ *is a basis for* V*.*

Proposition .4. *If* V *is a vector space and* S *is a linearly independent subset but* $\operatorname{span} S \neq V$*, then some set* S' *containing* S *is a basis.*

Proposition .5. *If* V *is known to have dimension* n *and* S *has* n *members and is linearly independent, then* $\operatorname{span} S = V$ *and so* S *is a basis for* V*.*

Proposition .6. *If* V *is known to have dimension* n *and* S *has* n *members and* $\operatorname{span} S = V$*, then* S *is linearly independent and so* S *is basis for* V*.*

Proof of Proposition .3. Suppose $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset V$ is linearly dependent. Then for some constants, not all zero, we have

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}. \tag{1}$$

Assume without loss of generality that $c_j \neq 0$. Let $S_1 = S \backslash \{v_j\}$, that is we delete $\mathbf{v}_j$ from S to form S_1. Since $c_j \neq 0$, we can solve equation (1) for $\mathbf{v}_j$.

$$\mathbf{v}_j = \sum_{i \neq j} \frac{c_i}{cj}\mathbf{v}_i.$$

Thus, $\mathbf{v}_j \in \operatorname{span} S_1$. This implies that $\operatorname{span} S_1 = \operatorname{span} S$. (Prove this.)

If S_1 is linearly independent it is a basis for V. If it is not we can repeat the above process, deleting a redundant vector from S_1 forming a new set S_2. If S_2 is linearly independent, we are done. If not we continue. Eventually we will have an S_k with only one member. Since S did not contain the zero vector, S' must be linearly independent. ☐

Proof of Proposition .4. ☐

Problem 2. Prove Proposition .5.

Problem 3. Prove Proposition .6.

Example 3. Let $S = \{\sin^2 x, \cos^2 x, \cos 2x, 5\}$. Let $V = \operatorname{span} S$. The set S is linearly dependent. Since, $\cos 2x = \cos^2 x - \sin^2 x$ and $5 = 5(\sin^2 x + \cos^2 x)$, we can see that $\{\sin^2 x, \cos^2 x\}$ spans V. The reader can check that this set is linearly independent and thus is a basis for V.

Problem 4. Is the set $\{\tan^2 x, \sec^2 x, 1\}$ linearly independent? If not find a linearly independent subset with the same span.

Problem 5. Is the set $\{\sin x, \sin 3x, \sin^3 x\}$ linearly independent? If not find a linearly independent subset with the same span.

Example 4. Let V equal the range of $A = \begin{bmatrix} 2 & 3 & 4 & 3 & 3 \\ 1 & 1 & 1 & 2 & 0 \\ 2 & 1 & 1 & 3 & 1 \end{bmatrix}$. Remember that this is just the span of the column vectors of A. Find a basis for V.

Solution. ☐

Problem 6. Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} \right\}$. Let $V = \operatorname{span} S$. Show that S is linearly dependent. Find a subset of S that is a basis for V. What is the dimension of V?

Problem 7. Find a basis for $\mathbb{R}^3$ that contains the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Need to add more problems.