Chapter 4

Determinants

In this chapter all matrices are assumed to be square unless stated otherwise.

1 Executive Summary

We present here just the bare bones of what students need to know about the determinant of a square matrix to apply and compute them. The non-diligent reader can, after this section, skip to Section 11 on eigenvalues and eigenvectors.

1.1 Definitions and computations

We will define the **determinant** of a square $(n \times n)$ matrix. Two notations are commonly used for the determinate of a square matrix A:

$$\det(A) \qquad \text{and} \qquad |A|$$

Don't confuse $|A|$ with the absolute value. Determinants can be negative.

Definition 4.1. For 2×2 matrices we define $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to be $ad - bc$.

If A is an $n \times n$ matrix, let A_{ij} be the $(n-1) \times (n-1)$ matrix formed by deleting the i^{th} row and j^{th} column of A. Then define the determinant of A by,

$$|A| = a_{11}|A_{11}| - a_{21}|A_{21}| + a_{31}|A_{31}| \cdots \pm a_{n1}|A_{n1}|$$

$$= \sum_{i=1}^{n} (-1)^{i+1} a_{i1} |A_{i1}|.$$

This again is called *expansion along the first column.*

Example 1. $\begin{vmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 3 & -1 \end{vmatrix} = 2 \begin{vmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} - \begin{vmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 3 & -1 \end{vmatrix} +$

$$0 \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix} - (-1) \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 2 \times 3 - 9 + 0 + 4 = 1.$$

Problem 1. Find the determinants for the two matrices below by expanding along the first column.

$$A = \begin{bmatrix} 4 & -4 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 4 \\ 0 & -3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & 1 & 2 & 1 \\ 2 & -2 & 5 & 2 \\ 3 & 3 & 0 & 0 \end{bmatrix}$$

Answers: $\det(A) = 75$, $\det(B) = -135$.

It can be shown that the determinant can be commputed just as well by expanding along any row or column where we adjust the signs as follows.

Theorem 4.2. *Let A be an $n \times n$ matrix. Then*

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{i1} |A_{ij}|$$

and

$$\det(A) = \sum_{i=j}^{n} (-1)^{i+j} a_{i1} |A_{ij}|.$$

In the first sum j is a fixed number bewteen 1 and n that determines a fixed column of A. In the second sum i is a fixed number bewteen 1 and n that determines a fixed row of A.

1.2 Effects of row & column operations

1. If A' is derived from A by multiplying a row or column by a real number r, then $\det(A') = r\det(A)$.
2. If A' is derived from A by adding a multiple of one row to another row or a multiple of one column to another column then $\det(A') = \det(A)$
3. If A' is derived from A by switching two rows or by switching two columns then $\det(A') = -\det(A)$.

An efficient method for computing determinants is to use row operations to convert A into an upper triangluar matrix (one with all 0's below the diagonial). Then the determinant is just the product of the diagonal elements.

1.3 Important properies

1. $\det(AB) = \det(A)\det(B)$.
2. $\det(A^T) = \det(A)$.
3. A has an inverse if and only if $\det(A) \neq 0$.
4. If A^{-1} exists then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.
5. The rows and columns of A are linearly inependent if and only if $\det A \neq 0$.

2 Introduction

We will define the **determinant** of a square ($n \times n$) matrix. Two notations are commonly used for the determinate of a square matrix A:

$$\det(A) \qquad \text{and} \qquad |A|$$

It turns out that it is more natural to think of the determinant as a function whose input is n vectors from $\mathbb{R}^n$ – *e.g.* the row vectors of an $n \times n$ matrix – that outputs a real number. The essential nature of the determinant function is that it gives a way, and in some sense the only way, to define volume in spaces with dimension greater than three.

By using certain properties of the determinant we will develop some short cuts for computing determinants. These properties will also lead to important applications of the determinant function.

This section will move between three different levels of abstraction:

- computation of determinants,
- properties of the determinant function, and
- the essential nature of the determinant function as a volume function.

Pay careful attention to the interaction between these levels.

It is advisable to skip the proofs on the first reading. Some instructors may wish to skip some of the harder proofs altogether. If fact, there are several theorems in this chapter whose proofs are not give at all or are just outlined. These proofs are usually covered in more advanced courses.

3 Determinants for n = 1, 2 and 3

Let $n = 1$. Then we define $\det([a])$ to be a. Geometrically the is the length of the vector $\langle a \rangle$ in $\mathbb{R}^1$, except that it can be negative: $\det([-3]) = -3$. The sign of $\det([a])$ tells us the direction the vector points. We say $\det([a])$ is the *signed length* or the *oriented length* of the vector.

Let $n = 2$. Recall from Section 2 that for a 2×2 matrix $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$. Recall also that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ was the signed area of the parallelogram determined by the row vectors, $[a\, b]$ and $[c\, d]$. (If you did not read Section 2 before, now would be a good time to at least skim through it.) Here are three properties of the determinant of 2×2 matrices. Their proofs are left to the you. Each has a geometric interpretation.

1. If A' is obtained from A by switching the rows, then $\det(A') = -\det(A)$.
2. If A' is obtained from A by multiplying either row by a real number c, then $\det(A') = c\det(A)$. Notice that this means $\det(cA) = c^2 \det(A)$.
3. If A' is obtained from A by adding a multiple of one row to the other, then $\det(A') = \det(A)$. This may seem surprising at first.

Problem 1. Prove the three properties of 2×2 determiants listed above.

Property 1 says that taking the mirror image of a parallelogram changes its "orientation". Property 2 says that stretching a parallelogram in the direction of one edge by a factor of c, changes the area by a factor of c. Notice that if we stretch both edges by c the area changes by a factor of c^2.

Now for Property 3. What is it trying to tell us? The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ determines the rectangle $[1\,0] \times [0\,2]$, which has area $\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = 2$. Now add three times the first row to the second to get $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$. See Figure 4.1. Notice the new parallelogram still has base one and height two. Thus the areas are equal, as are the determinants! Property 3 says, loosely, that sliding a parallelogram along the direction of one of its edges does not change the area.

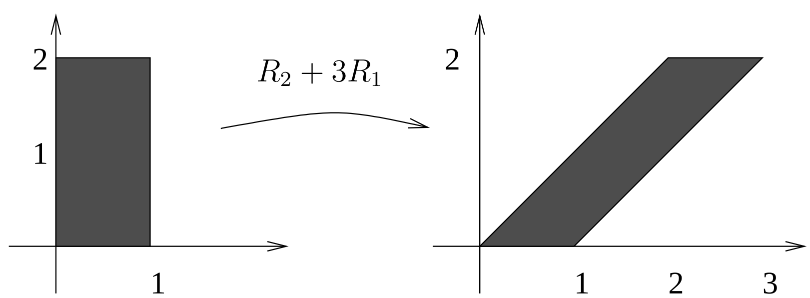

Figure 4.1: Area is unchanged.

Finally we recall from Section 2 that for 2×2 matrices $\det(AB) = \det(A)\det(B)$, A is invertible if and only if $\det(A) \neq 0$, and $\det(A^T) = \det(A)$. Later in this section we will show that these statements hold for all $n \times n$ matrices.

On to $n = 3$. We define the determinant of a 3×3 matrix by the formula below.

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - d \begin{vmatrix} b & b \\ h & i \end{vmatrix} + g \begin{vmatrix} b & c \\ e & f \end{vmatrix}.$$

This is called **expansion along the first column.**

Problem 2. Compute the following determinants.

a. $\begin{vmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{vmatrix}$ b. $\begin{vmatrix} 2 & 3 & -2 \\ 0 & 5 & 8 \\ 0 & 0 & 7 \end{vmatrix}$ c. $\begin{vmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \\ -1 & 3 & 0 \end{vmatrix}$.

The determinant of a 3×3 matrix does give $\pm$ the volume of the parallelepiped determined by its row vectors. Let's look at an easy example. Consider 3×3 diagonal matrices. In this case $\det \begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{bmatrix} = aei$ which is the signed volume of the rectangular prism $[a, 0, 0] \times [0, e, 0] \times [0, 0, i]$.

Problem 3. Prove that the determinant of a 3×3 matrix is $\pm$ the volume of the parallelepiped determined by its row vectors. (This can be found in most third semester calculus textbooks.)

The effects of performing row operations on 3×3 matrices is the same as for 2×2 matrices.

Problem 4. Prove the following statements for 3×3 matrices.
a. If A' is obtained from A by switching any two rows, then $\det(A') = -\det(A)$.
b. If A' is obtained from A by multiplying any row by a real number c, then $\det(A') = c \det(A)$. Notice that this means $\det(cA) = c^3 \det(A)$.
c. If A' is obtained from A by adding a multiple of one row to another, then $\det(A') = \det(A)$. (This is hard and is done in the next section.)
Isn't this just what a volume function should do?

Problem 5. Let A be a 3×3 matrix.
a. What can you say about the determinant if two rows are the same?
b. What can you say about the determinant if A has a row of zeros?
c. Prove $\det(A) = \det(A^T)$. (This shows that the parallelepiped determined by the rows of A has the same oriented volume as the one determined by the columns of A.)
d. What can you say about the determinant if two column are the same? What can you say about the determinant if A has a column of zeros?
e. Prove that if B is also a 3×3 matrix then $\det(AB) = \det(A)\det(B)$.
f. Prove that A is invertible (nonsingular) if and only if $\det(A) \neq 0$.
g. Prove that if A is invertible then $\det(A^{-1}) = 1/\det(A)$. Hint: what is $\det(I)$?

Example 1. Find $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{vmatrix}$. *Solution.* Subtract row 1 from row two. This does not change the value of the determinant. Since we now have two identical rows, the determinant must be zero.

4 Recursive Definition of the Determinant

Now we define, recursively, the determinant of an $n \times n$ matrix.

Definition 4.3. If A is an $n \times n$ matrix, let A_{ij} be the $(n-1) \times (n-1)$ matrix formed by deleting the i^{th} row and j^{th} column of A. Then define the determinant of A by,

$$|A| = a_{11}|A_{11}| - a_{21}|A_{21}| + a_{31}|A_{31}| \cdots \pm a_{n1}|A_{n1}|$$

$$= \sum_{i=1}^{n} (-1)^{i+1} a_{i1}|A_{i1}|.$$

This again is called *expansion along the first column.*

Example 1. $\begin{vmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 3 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} - \begin{vmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 3 & -1 \end{vmatrix} +$

$$0\begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 2 \times 3 - 9 + 0 + 4 = 1.$$

Problem 1. Find the determinants for the two matrices below by expanding along the first column.

$$A = \begin{bmatrix} 4 & -4 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 4 \\ 0 & -3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & 1 & 2 & 1 \\ 2 & -2 & 5 & 2 \\ 3 & 3 & 0 & 0 \end{bmatrix}$$

Answers: $\det(A) = 75$, $\det(B) = -135$.

Next we establish two lemmas – a lemma is a small theorem that is used to prove a larger theorem. These two lemmas will be used in the next section. The first one is interesting is its own right.

Lemma 4.4. *Let A be an $n \times n$ matrix and let A' be obtained from A by switching any two rows. Then* $\det(A') = -\det(A)$.

Partial Proof based on Len Evans notes. **Step 1.** First we consider a special case. Suppose A' is obtained from A by switching two adjacent rows. We will show $\det(A') = -\det(A)$. We will assume that this is true for 3×3 matrices (Problem 4a), and just work out the 4×4 case. To simplify notation we will switch the second and third rows of a 4×4 matrix. Other adjacent row switchings are done in the same way.

Let

$$A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} \text{ and } A' = \begin{bmatrix} R_1 \\ R_3 \\ R_2 \\ R_4 \end{bmatrix}.$$

Now

$$|A'| = a'_{11}|A'_{11}| - a'_{21}|A'_{21}| + a'_{31}|A'_{31}| - a'_{41}|A'_{41}|. \qquad (\#)$$

Notice that

$$a'_{11} = a_{11} \text{ and } a'_{41} = a_{41} \text{ while } a'_{21} = a_{31} \text{ and } a'_{31} = a_{21}.$$

That is, $a'_{j1} = a_{j1}$, unless j corresponds to one of the rows being switched, in which case they switch with the rows. Also notice that if j is not one of the row being switched then A'_{j1} can be obtained from A_{j1} by switching two adjacent rows. Since these are 3×3 we have

$$|A'_{11}| = -|A_{11}| \text{ and } |A'_{41}| = -|A_{41}|$$

What happens to $|A'_{21}|$ and $|A'_{31}|$? We can check that $A'_{21} = A_{31}$ and $A'_{31} = A_{21}$. (This is where the adjacency assumption is needed.) It follows that

$$|A'_{21}| = |A_{31}| \text{ and } |A'_{31}| = |A_{21}|$$

Next we apply what we have learned to equation (#) to get

$$|A'| = -a_{11}|A_{11}| - a_{31}|A_{31}| + a_{21}|A'_{21}| + a_{41}|A_{41}| =$$

$$-1(a_{11}|A_{11}| - a_{21}|A_{21}| + a_{31}|A'_{31}| - a_{41}|A_{41}|) = -|A|.$$

The $n \times n$ case can be done by induction.

Step 2. Now we drop the assumption that the rows to be switched are adjacent. Suppose A is an $n \times n$ matrix and A' is obtained from A by switching rows j and k, with $j < k$.

By Problem 5 in Section 3 of Chapter 0 (page 15) this switching can be decomposed into an odd number of adjacent row switchings. But, -1 raised to an odd power is -1. Thus, $|A'| = -|A|$. □

Lemma 4.5. *Let A, A' and A'' be $n \times n$ matrices that differ only in their k-th rows and suppose that the k-th row of A is the sum of the k-th rows of A' and A''. Then*

$$\det(A) = \det(A') + \det(A'').$$

Proof of 3×3 *case.* The lemma is easy to check for the 2×2 case. Let,

$$A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix},\ A' = \begin{bmatrix} R_1 \\ R_2 \\ R_3' \end{bmatrix} \text{ and } A'' = \begin{bmatrix} R_1 \\ R_2 \\ R_3'' \end{bmatrix},$$

where $R_3 = R_3' + R_3''$. The proofs for splitting rows 1 or 2 are similar. By expanding along the first column we get

$$|A| = a_{11}|A_{11}| - a_{21}|A_{21}| + a_{31}|A_{31}| \qquad (*)$$

and

$$|A'|+|A''| = a'_{11}|A'_{11}|-a'_{21}|A'_{21}|+a'_{31}|A'_{31}|+a''_{11}|A''_{11}|-a''_{21}|A''_{21}|+a''_{31}|A''_{31}|. (**)$$

We want to show these two equations are equal. Of course we have

$$a_{11} = a'_{11} = a''_{11},\ \ a_{21} = a'_{21} = a''_{21} \text{ and } a_{31} = a'_{31} + a''_{31}.$$

But also, since the lemma holds for 2×2 matrices,

$$|A_{11}| = |A'_{11}| + |A''_{11}| \text{ and } |A_{21}| = |A'_{21}| + |A''_{21}|.$$

Finally, when we delete the third row we get

$$|A_{31}| = |A'_{31}| = |A''_{31}|,$$

because $A_{31} = A'_{31} = A''_{31}$.

We apply these to equation (∗).

$$|A| = a_{11}(|A'_{11}| + |A''_{11}|) - a_{21}(|A'_{21}| + |A''_{21}|) + (a'_{31} + a''_{31})|A_{31}| =$$

$$a_{11}|A'_{11}| + a_{11}|A''_{11}| - a_{21}|A'_{21}| - a_{21}|A''_{21}| + a'_{31}|A_{31}| + a''_{31}|A_{31}| =$$

$$a'_{11}|A'_{11}| + a''_{11}|A''_{11}| - a'_{21}|A'_{21}| - a''_{21}|A''_{21}| + a'_{31}|A'_{31}| + a''_{31}|A''_{31}|.$$

By rearranging these terms we can use equation (∗∗) to get $|A| = |A'| + |A''|$. The general proof uses induction. □

5 The Determinant as a Volume Function

If the determinant is to serve as an oriented volume function for each $\mathbb{R}^n$, then it should have at least these three properties.

1. $\det(I) = 1$; that is, the unit cube should have volume one in all dimensions.

2. If A' is obtained from A by multiplying a row of A by a real number c, then $\det(A') = c\det(A)$ – the stretching property.

3. If A' is obtained from A by adding one row of A to another, then $\det(A') = \det(A)$ – the sliding property.

Not only does the determinant function satisfy these three properties, it is the only function to do so!

Theorem 4.6 (The Fundamental Theorem of Determinants)**.** *The determinate is the unique function that satisfies the three properties above.*

We will not prove Theorem 4.6 in its entirety. The uniqueness part is beyond the scope of this course. We shall merely outline the proof that determinant function satisfies the three properties.

Partial Proof. 1. The first property should be obvious, but we could do it by induction on n where I_n is the $n \times n$ identity matrix.

2. You should have worked out the 2×2 and 3×3 cases (Problems 1 and 4b respectively). We will do the 4×4 case in such a way that you should be able to see how to get the general case by induction. Let $A = [a_{ij}]$ be a 4×4 matrix. Multiply the first row by c to obtain A'.

Now,

$$|A'| = ca_{11}|A'_{11}| - a_{21}|A'_{21}| + a_{31}|A'_{31}| - a_{41}|A'_{41}|.$$

Since A'_{11} is the same as A_{11} (A' only differs from A in the first row), we have $|A'_{11}| = |A_{11}|$.

Next notice that A'_{21} can be obtained from A_{21} by multiplying the first row of A_{21} by c. Thus, $|A'_{21}| = c|A_{21}|$, since we know Property 2 holds for 3×3 matrices. Likewise, $|A'_{31}| = c|A_{31}|$ and $|A'_{41}| = c|A_{41}|$. We can now write

$$|A'| = ca_{11}|A_{11}| - a_{21}c|A_{21}| + a_{31}c|A_{31}| - a_{41}c|A_{41}|.$$

If we factor out the c then we get $|A'| = c|A|$. The argument is similar for any row of A. This establishes Property 2 for 4×4 matrices. Hopefully you can see how one could use the Principle of Mathematical Induction to get the general result.

3. Property 3 is the hardest to prove. Here is where we use Lemmas 4.4 and 4.5. Property 3 is easy to verify for 2×2 matrices (Problem 1). We assume this and do the 3×3 case. (The 3×3 case was done in Problem 4c. But the proof here is set up so that you can see how it might be generalized by induction for $n \times n$ matrices.) Let

$$A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} R_1 \\ R_2 \\ R_3 + R_1 \end{bmatrix}$$

Other row additions are done similarly. Now then,

$$|A| = a_{11}|A_{11}| - a_{21}|A_{21}| + a_{31}|A_{31}| \qquad (!)$$

and

$$|A'| = a'_{11}|A'_{11}| - a'_{21}|A'_{21}| + a'_{31}|A'_{31}|. \qquad (!!)$$

We will transform (!!) into (!). We have

$$a'_{11} = a_{11}, \ \ a'_{21} = a_{21} \text{ and } a'_{31} = a_{31} + a_{11}.$$

Also, $A'_{31} = A_{31}$ implies $|A'_{31}| = |A_{31}|$ and $|A'_{21}| = |A_{21}|$ follows from the 2×2 case. Now,

$$A'_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} + a_{12} & a_{33} + a_{13} \end{bmatrix}.$$

By Lemma 4.5

$$|A'_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{12} & a_{13} \end{vmatrix} =$$

$$|A_{11}| - |A_{31}|.$$

The fact that $\begin{vmatrix} a_{22} & a_{23} \\ a_{12} & a_{13} \end{vmatrix} = -|A_{31}|$ comes from Lemma 4.4 on row switching. We apply what we know to equation (!!) to get

$$|A'| = a_{11}(|A_{11}| - |A_{31}|) - a_{21}|A_{21}| + (a_{31} + a_{11})|A_{31}| =$$

$$a_{11}|A_{11}| - a_{21}|A_{21}| + a_{31}|A_{31}| = |A|.$$

This completes our outline of the proof. ☐

6 More Properties of the Determinant Function

Theorem 4.7 (The Effect of Row Operations)**.** *1. If A' is obtained from A by switching two rows, then* $\det A' = -\det A$.

2. If A' is obtained from A by multiplying a row by a constant c, then $\det A' = c \det A$.

3. If A' is obtained from A by adding the multiple of one to another then $\det A' = \det A$.

Proof. Statement 1 is just Lemma 4.4. Statement 2 is Property 2 in Theorem 4.6. Statement 3 follows from Properties 2 and 3 in Theorem 4.6 as we now show. Let A_0 be an $n \times n$ matrix and let A_3 be obtained from A_0 by adding c times row j to row k; assume with out loss of generality that $c \neq 0$. Consider the following matrices.

$$A_0 = \begin{bmatrix} R_1 \\ \vdots \\ R_j \\ \vdots \\ R_k \\ \vdots \\ R_n \end{bmatrix}, A_1 = \begin{bmatrix} R_1 \\ \vdots \\ cR_j \\ \vdots \\ R_k \\ \vdots \\ R_n \end{bmatrix}, A_2 = \begin{bmatrix} R_1 \\ \vdots \\ cR_j \\ \vdots \\ R_k + cR_j \\ \vdots \\ R_n \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} R_1 \\ \vdots \\ R_j \\ \vdots \\ R_k + cR_j \\ \vdots \\ R_n \end{bmatrix}.$$

Then we have

$$\begin{aligned} |A_1| &= c|A_0| && \text{by Statement 2} \\ |A_2| &= |A_1| && \text{by Property 3 of Theorem 4.6} \\ |A_3| &= \tfrac{1}{c}|A_2| && \text{by Statement 2} \end{aligned}$$

Thus, $|A_3| = \frac{1}{c}c|A_0| = |A_0|$. □

Theorem 4.7 gives a tool for simplifying the computing of determinants. We use row operations to transform A into an upper triangular matrix, tracking how the row operation effect the determinant. The determinant of an upper triangular matrix is just the product of its diagonal entries. From this information we can deduce the determinant of A.

Example 1 (Upper triangular matrices)**.** In the following example (from Len Evans', *A Brief Course in Linear Algebra*) identify the operation being used.

$$\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 3 & 0 & 1 & 1 \\ -1 & 6 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & -6 & 4 & -2 \\ 0 & 8 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & -5 & -5 \end{vmatrix} =$$

$$-5\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 1 & 1 \end{vmatrix} = +5\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 7 & 4 \end{vmatrix} = +5\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{vmatrix}.$$

The last matrix is an upper triangular matrix. Its determinant is especially easy to compute.

$$\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{vmatrix} = 1 \cdot 2 \cdot \begin{vmatrix} 1 & 1 \\ 0 & -3 \end{vmatrix} = 1 \cdot 2 \cdot 1 \cdot (-3) = -6.$$

Thus the determinant of the original matrix is $5 \cdot (-6) = -30$.

Problem 1. a. Prove that if a square matrix has two identical rows its determinant is zero. b. Prove that if a square matrix has a row of zeros its determinant is zero.

7 Two Big Theorems

Theorem 4.8. A^{-1} *exists if and only if* $\det A \neq 0$.

Theorem 4.9. $\det AB = \det A \det B$.

Remark. Both of these can be made intuitively plausible by thinking in terms of volume.

Proof of Theorem 4.8. The square matrix A is nonsingular (i.e. invertible) if and only if there exists a sequence of row operations taking A to I.

If $\det A = 0$ any matrix derived from A by row operations will also have zero determinant. Hence A is not row equivalent to I and so A^{-1} does not exist.

Suppose now that A is known to be not invertible. Let $B = \operatorname{rref}(A)$ Then B cannot have a complete set of pivots, that is B must have a zero on its diagonal. But B is an upper triangular matrix (because it is reduced). Thus $\det B = 0$, which implies $\det A = 0$. □

Proof of Theorem 4.9. First we establish the following fact. Let A and B be matrices such that B can be derived from A by a single row operation which we denote by r, i.e.

$$A \xrightarrow{r} B.$$

Now let C be a third matrix and consider the products AC and BC. Our claim is that if you apply the same row operation r to AC you get BC,

$$AC \xrightarrow{r} BC.$$

I will show this for row operation 3 for 3×3 matrices. You should be able to see how this could be extended to cover all the other cases.

Let

$$A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}, \qquad B = \begin{bmatrix} R_1 \\ R_2 \\ R_3 + kR_2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}.$$

Then

$$AC = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix}$$

and

$$BC = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ (R_3+kR_2)C_1 & (R_3+kR_2)C_2 & (R_3+kR_2)C_3 \end{bmatrix} =$$

$$\begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1+kR_2C_1 & R_3C_2+kR_2C_2 & R_3C_3+kR_2C_3 \end{bmatrix}.$$

Thus original row operation takes AC to BC. We are now in position to prove Theorem 2. The prove is divided into two cases.

Case 1: Suppose A is nonsingular ($\det A \neq 0$). Then there are k row operations taking A to I, for some number k:

$$A = A_0 \xrightarrow{r_1} A_1 \xrightarrow{r_2} A_2 \xrightarrow{r_3} \cdots \xrightarrow{r_k} A_k = I$$

For each r_i there is a nonzero number c_i (which could be 1) such that $\det A_{i-1} = c_i \det A_i$. Thus,

$$\det A = c_1 \det A_1 = c_1c_2 \det A_2 = \cdots = c_1c_2c_3\cdots c_k \det I,$$

and so we can write $\det A = c_1c_2\cdots c_k$. Now apply exactly the same row operations to the product AB.

$$AB \xrightarrow{r_1} A_1B \xrightarrow{r_2} A_2B \xrightarrow{r_3} \cdots \xrightarrow{r_k} IB.$$

Thus we have

$$\det AB = c_1\cdots c_k \det B = \det A \det B.$$

Case 2: Suppose $\det A = 0$. We must show that $\det AB = 0$ since $\det A \det B = 0 \cdot \det B = 0$.

Since A is not invertible there is a sequence of row operations taking A to a matrix Z that has a row of zeros. (Why? A good test question!?)

$$A = A_0 \xrightarrow{r_1} A_1 \xrightarrow{r_2} A_2 \xrightarrow{r_3} \cdots \xrightarrow{r_k} A_k = Z.$$

Thus,

$$AB \xrightarrow{r_1} A_1B \xrightarrow{r_2} A_2B \xrightarrow{r_3} \cdots \xrightarrow{r_k} ZB.$$

Then

$$\det AB = c_1\cdots c_k \det ZB.$$

But if Z has a row of zeros so does ZB (check this!). Thus, $\det ZB = 0$. This completes our proof. □

8 Transposes and another short cut

Theorem 4.10 (Transposes)**.** *$\det A^T = \det A$.*

Proof. Need to know that A^T is invertible iff A is. [This is a big gap. We need det AB = detA detB and facts about elementary matrices.] □

We can now produce more tools for calculating determinants.

Theorem 4.11. *1. Statements (1-3) in Theorem 4.7 remain true if "rows" is replaced with "columns".*
2. We can compute a determinant by expanding along any row or column, provided we watch our signs.

Outline of proof. Statement 1 follows from Theorem 4.10 and Theorem 4.7. For Statement 2 just use row switching and transposes to place the row or column you wish to expand along in the first column. Of course you have to watch how any row switching effects the signs. □

Example 1. We show that the determinant of a 3×3 matrix can be obtained by expand along the second row.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ h & i & j \end{vmatrix} = -\begin{vmatrix} d & e & f \\ a & b & c \\ h & i & j \end{vmatrix} = -\begin{vmatrix} d & a & g \\ e & b & h \\ f & c & i \end{vmatrix} =$$

$$-\left(d\begin{vmatrix} b & h \\ c & i \end{vmatrix} - e\begin{vmatrix} a & g \\ c & i \end{vmatrix} + f\begin{vmatrix} a & g \\ b & h \end{vmatrix}\right) =$$

$$-d\begin{vmatrix} b & h \\ c & i \end{vmatrix} + e\begin{vmatrix} a & g \\ c & i \end{vmatrix} - f\begin{vmatrix} a & g \\ b & h \end{vmatrix}.$$

This is the same as expanding along the second row, but with the signs switched. In general the "checkerboard" patterns below tells us how the signs go.

$$n \text{ odd: } \begin{vmatrix} + & - & + & \cdots & - & + \\ - & + & - & \cdots & + & - \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ + & - & + & \cdots & - & + \end{vmatrix}, \quad n \text{ even: } \begin{vmatrix} + & - & + & \cdots & + & - \\ - & + & - & \cdots & - & + \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ - & + & - & \cdots & + & - \end{vmatrix}.$$

Problem 1. Repeat the determinant calculations for the matrices A and B given in Problem 1 by expanding along different rows and columns.

9 Cramer's Rule [Optional]

Cramer's Rule is another method for solving an $n \times n$ system of linear equations. It is based on another way of finding the inverse of a matrix. For larger systems it is an inefficient method. Row reduction to an upper triangular matrix is best.

However, if the matrix entries are variables or functions row reduction by hand may be messy. Cramer's Rule is often used in engineering and physics courses but is rarely used in math or computer science courses. We shall skip the proofs.

Definition 4.12. Given an $n \times n$ matrix A the ij **cofactor** is given by

$$c_{ij} = (-1)^{i+j} \det(A_{ij}),$$

where we recall that A_{ij} was obtained from A by deleting row i and column j.

The **adjoint matrix** of A is the matrix formed from its cofactors.

$$\text{adj}\,(A) = [c_{ij}]$$

Theorem 4.13. *If* $\det(A) \neq 0$ *then*

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)}.$$

Problem 1. Use Theorem 4.13 to find the inverses of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{and } B = \begin{bmatrix} x & 2x & 3 \\ x^2 & 3 & 6x \\ 2-x & x & 0 \end{bmatrix}$$

Answers.

$$A^{-1} = \begin{bmatrix} \dfrac{d}{a\,d - b\,c} & \dfrac{-b}{a\,d - b\,c} \\ \dfrac{-c}{a\,d - b\,c} & \dfrac{a}{a\,d - b\,c} \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} \dfrac{2x^2}{D} & \dfrac{-x}{D} & \dfrac{-4x^2-3}{D} \\ \dfrac{2x(x-2)}{D} & \dfrac{2-x}{D} & \dfrac{x^2}{D} \\ \dfrac{x^3+3x-6}{-3D} & \dfrac{x(3x-4)}{3D} & \dfrac{x(2x^2-3)}{3D} \end{bmatrix}$$

where $D = 5x^3 - 8x^2 + 6 - 3x$. □

Theorem 4.14 (Cramer's Rule)**.** *Let A be an $n \times n$ nonsingular matrix and let $\mathbf{b}$ be an $n \times 1$ column vector. Define $A(i, \mathbf{b})$ to be the matrix obtained by replacing the i-th column of A with $\mathbf{b}$. Suppose $\mathbf{v}$ is the solution to $A\mathbf{v} = \mathbf{b}$. We can compute the i-th entry of $\mathbf{v}$ by*

$$v_i = \frac{\det A(i, \mathbf{b})}{\det A}$$

Problem 2. Use Cramer's Rule to solve each system below.

(a) $$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 9 \end{bmatrix}$$

(b) $$\begin{bmatrix} x & 2x & 3 \\ x^2 & 3 & 6x \\ 2-x & x & 0 \end{bmatrix} \begin{bmatrix} f(x) \\ g(x) \\ h(x) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Answers. (a) $x = 1$, $y = 1$, and $z = 2$.
(b) Let $d(x) = 5x^2 - 3x - 6$. Then $f(x) = 2x/d(x)$, $g(x) = (2x - 4)/d(x)$ and $h(x) = -\frac{1}{3}(x^2 - 5x + 6)/d(x)$. □

10 Permutations Definition [Optional]

A **permutation** is a function that rearranges the elements of a list of numbers. Thus, $p : (1, 2, 3, 4, 5) \to (3, 2, 1, 5, 4)$ and $r : (1, 2, 3, 4, 5) \to (1, 2, 3, 5, 4)$ are permutations. Permutations of the same list can be composed: $p \circ r : (1, 2, 3, 4, 5) \to (3, 2, 1, 4, 5)$.

The set of all possible permutations of the list $(1, 2, 3, \ldots, n)$ will be called P_n.

Problem 1. List the elements of P_n for n equal to 2, 3, and 4.

Problem 2. Prove that P_n has $n!$ members. Hint: use induction.

The simplest permutation is one that just switches two entries in a list. Every permutation can be broken down into a sequence of switches. For example

$$(1, 2, 3, 4, 5) \to (3, 2, 1, 4, 5) \to (3, 2, 1, 5, 4)$$

and

$$(1,2,3,4,5) \to (2,1,3,4,5) \to (2,3,1,4,5) \to (2,3,1,5,4) \to (3,2,1,5,4)$$

are two ways to break p down into switches.

Problem 3. Consider $q : (1,2,3,4,5,6) \longrightarrow (3,4,2,1,5,6)$. Break q down into switches several different ways.

Problem 4 (Important)**.** Prove that for any given permutation the number of switches used to create it is either always even or always odd. Hints: use induction; compare with Problem 5 of Section 0.3.

The **parity** of a permutation is defined to be 0 if it decomposes into an even number of switches and 1 if it decomposes into an odd number of switches. The parity function is denoted by σ. Thus, $\sigma(p) = 0$ and $\sigma(q) = 1$.

We will be applying permutations to entries of matrices. For example if $A = [a_{ij}]^{6\times 6}$. then $a_{q(2)4} = a_{44}$ and $a_{q(1)q(6)} = a_{36}$.

We are now ready to present the alternative definition of determinants.

Definition 4.15. The determinant of an $n \times n$ matrix is given by

$$\det(A) = \sum_{p \in P_n} (-1)^{\sigma(p)} a_{1p(1)} a_{2p(2)} a_{3p(3)} \cdots a_{np(n)}$$

Check that for 3×3 matrices this becomes

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

Theorem 4.16. *Definitions 4.3 and 4.15 of the determinant are equivalent.*

We shall not do the proof.

Problem 5. Find the determinants of A and B give in Problem 1 in Section 2 using this alternative definition of a determinant.

Problem 6. Prove Theorem 4.16 for (a) 2×2 matrices, (b) 3×3 matrices, and (c) 4×4 mtrices.