Problem 1. Find and prove formulas for $\sum_{k=1}^{n} k^p$ for (a) $p = 4$, (b) $p = 5$, and (c) $p = 6$.

5 Simultaneous Systems of Equations

Material in this section will be useful in computing matrix inverses in Section 6. We proceed by examples.

Example 1. Let $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$, $C_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Solve

$$Ax = C_1 \quad \text{and} \quad Ax = C_2.$$

Solution 1. First we solve $Ax = C_1$.

$$\begin{bmatrix} 2 & 1 & | & 3 \\ 4 & 3 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & | & 3 \\ 0 & 1 & | & -5 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & | & 8 \\ 0 & 1 & | & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & 4 \\ 0 & 1 & | & -5 \end{bmatrix}$$

Now we work with $Ax = C_2$.

$$\begin{bmatrix} 2 & 1 & | & 0 \\ 4 & 3 & | & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & | & 0 \\ 0 & 1 & | & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & | & -2 \\ 0 & 1 & | & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & -1 \\ 0 & 1 & | & 2 \end{bmatrix}$$

Thus, we have,

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

□

You might notice that the row operations used in the two parts above are the same. It seems wasteful to go through the same steps twice.

Solution 2. Let $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ be a matrix of unknowns. Also let $B = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$. We shall solve $AX = B$. (Think hard to see why this is the same as the original problem.) We form the augmented matrix $\begin{bmatrix} A & | & B \end{bmatrix}$ and apply the same row operations.

$$\begin{bmatrix} 2 & 1 & | & 3 & 0 \\ 4 & 3 & | & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & | & 3 & 0 \\ 0 & 1 & | & -5 & 2 \end{bmatrix} \sim$$

$$\begin{bmatrix} 2 & 0 & | & 8 & -2 \\ 0 & 1 & | & -5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & 4 & -1 \\ 0 & 1 & | & -5 & 2 \end{bmatrix}$$

The conclusion is the same. □

Example 2. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$. Solve $AX = I$.

Solution.

$$\begin{bmatrix} 1 & 1 & | & 1 & 0 \\ 2 & 3 & | & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & | & 1 & 0 \\ 0 & 1 & | & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & 3 & -1 \\ 0 & 1 & | & -2 & 1 \end{bmatrix}$$

Thus $X = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$. Check this by doing the multiplication. □

Remark. For the above example check that $XA = I$ is also true. This is surprising!

Problem 1. Solve the follow simulatianious systems of equations, by hand, labeling all steps.

a. $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

6 Finding Inverses

Definition 2.6. Let A be a square matrix. If B is a matrix such that $AB = I$ then we say B is **a right inverse** of A. If B is a matrix such that $BA = I$ then we say B is **a left inverse** of A.

In the next section we establish these two facts.
(1) If $AB = I$ then $BA = I$, and vice versa, so left and right inverses are the same.
(2) If $AB = I$ and $AC = I$ then $B = C$, so inverses, when they exist, are unique.

Thus, we can rewrite the definition above as follows.

Definition 2.7. Let A be a square matrix. If there is a matrix B such $BA = AB = I$, we say B is **the inverse** of A, and denote it by A^{-1}.

We will develop a process that computes the inverse of a square matrix A, or shows that there is no inverse for A. Recall that the only real number that does not have a multiplicative inverse is zero. The situation for matrices is a bit different.

Given A, let X be a matrix of unknowns that is the same size as A. We wish to solve $AX = I$. But this is just a simultaneous systems of equations problem. If A is row equivalent to I we get a unique solution for X. If A is not row equivalent to I then it will turn out that there no solutions and A does not have an inverse.

Example 1. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. Show that A^{-1} exists and find it.

Solution. We set up the augmented matrix and reduce it.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & 4 & 2 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -3R_1 \\ -R_1 \end{array} \longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 2 & -3 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] R_2 \leftrightarrows R_3 \longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \\ 0 & -2 & 2 & -3 & 1 & 0 \end{array}\right] \begin{array}{l} \\ -1 \\ -2R_2 \end{array} \longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 4 & -1 & 1 & -2 \end{array}\right] \begin{array}{l} -2R_2 \\ \\ 1/4 \end{array} \longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1/4 & 1/4 & -1/2 \end{array}\right] \begin{array}{l} +2R_3 \\ -R_3 \\ \end{array} \longrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & 1/2 & 1 \\ 0 & 1 & 0 & 5/4 & -1/4 & -1/2 \\ 0 & 0 & 1 & -1/4 & 1/4 & -1/2 \end{array}\right] \longrightarrow$$

Thus,

$$A^{-1} = \begin{bmatrix} -3/2 & 1/2 & 1 \\ 5/4 & -1/4 & -1/2 \\ -1/4 & 1/4 & -1/2 \end{bmatrix}$$

□

Example 2. Let $A = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}$. Prove that A is not invertible.

Solution.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -3 & -6 & 0 & 1 \end{array}\right] \begin{array}{c} \\ +3R_1 \end{array} \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right]$$

We can see that there are no solutions. Thus A is not invertible. □

Problem 1. For each of the matrices below find the inverse or show no inverse exists.

$$\text{a. } \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \quad \text{b. } \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{c. } \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\text{d. } \begin{bmatrix} 3 & 2 & 1 \\ 2 & -1 & -1 \\ 1 & 4 & 0 \end{bmatrix} \quad \text{e. } \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Problem 2. For what values of x are the matrices below not invertible?

$$\text{a. } \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 7-x \end{bmatrix} \quad \text{b. } \begin{bmatrix} 2-x & 1 \\ 6 & 2 \end{bmatrix} \quad \text{c. } \begin{bmatrix} x+1 & 1 \\ -1 & x+1 \end{bmatrix}$$

Problem 3. Prove that a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$. Find a general formula for the inverse. This is a useful formula that you will want to remember.

Example 3. Determine whether these four points in $\mathbb{R}^3$ lie in a common plane: $(1, 2, 0)$, $(-1, -1, 5)$, $(0, 1, 3)$, $(7, 4, 12)$.

Outline of Solution. Substitute each of the four points into the equation $Ax + By + Cz - D = 0$. This gives four equations in four unknowns. Set this up as a matrix problem:

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ -1 & -1 & 5 & -1 \\ 0 & 1 & 3 & -1 \\ 7 & 4 & 12 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Clearly $A = B = C = D = 0$ is a solution, and would be a solution no matter which points we were given. But, this trivial solution just gives the equation $0 = 0$ whose solution set is all of $\mathbb{R}^3$. To get a plane that contains the four points requires the existence of nontrivial solutions. If the matrix is invertible, the solution is unique and thus $A = B = C = D = 0$ is the only solution; in this case the points do not lie in a single plane. If the matrix is noninvertible, then we will get infinitely many solutions – this system cannot be inconsistent because the output vector is all zeros. Each nontrivial solution corresponds to an equation for a plane containing the four points.

However, there are not necessarily infinitely many planes. Remember, any plane can be represented by infinitely many equations. But, it is possible that there are infinitely many planes that work. This would happen if the given points were on the same line. How could you tell this from the form of the solution set for A, B, C and D? □

Problem 4. Complete Example 3.

Definition 2.8. An invertible matrix is called **nonsingular**. A noninvertible matrix is called **singular**. This terminology may seem unnatural to you now but it is standard.

7 Properties of Inverses

Theorem 2.9. *We will establish the following:*

1. *If the inverse of A exists, it is unique.*
2. *If $AB = I$ then $BA = I$.*
3. *If A and B are invertible then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.*

4. *If A is invertible then so is A^{-1}, and $(A^{-1})^{-1} = A$.*

5. *If A is invertible then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.*

Remark. The proofs of 1 and 2 are a bit subtle. Some instructors may choose to skip over them. Everyone ought to be able to understand the proofs of 3, 4 and 5.

Proof of 1. Let A be an $n \times n$ matrix. Suppose A is row equivalent to I. Hence, the inverse of A exists and is unique. Suppose then, that A is not row equivalent to I.

A priori, it could be that $AX = I$ has infinitely many solutions, but we will show it has none. Suppose that when we put $[A|I]$ into a reduced row echelon form we get $[A'|B]$. We know that A' has a row of zeros since otherwise it would have n pivots and so would be equal to I. Thus, if $A'X = B$ has a solution, B must also have a row of zeros. (See Problem 1.) This would mean that I is row equivalent to a matrix with a row of zeros. But this is impossible. (See Problem 2.) Therefore, $A'X = B$ has no solutions, and hence $AX = I$ has no solutions.

Problem 1. a. Let A be a 5×5 matrix for which row three is all zeros. Let B be any 5×5 matrix. Prove that AB has a row of all zeros.
b. Let A be an $m \times n$ matrix with its k-th row all zeros. Let B be any $n \times p$ matrix. Prove that AB has a row of zeros.

Problem 2. a. Prove that the 4×4 identity matrix is not row equivalent to a matrix that has a row of zeros.
b. Prove that the $n \times n$ identity matrix is not row equivalent to a matrix that has a row of zeros.

Proof of 2. Suppose $AB = I$, where A and B are $n \times n$ square matrices. Thus A is row equivalent to I. Let X be an $n \times n$ matrix of unknowns. Then, $AX = A$ has a unique solution. Clearly that solution is $X = I$. But obverse: $A(BA) = (AB)A = IA = A$. Thus, $X = BA$ is also a solution to $AX = A$. Since the solution is unique, it must be that $I = BA$.

Proof of 3. Let X be an $n \times n$ matrix of unknowns. To find the inverse of AB we only need to find a solution to $ABX = I$. Let $X = B^{-1}A^{-1}$. Then $ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$. Thus, $X = B^{-1}A^{-1}$ is the unique inverse of AB.

Intuitive "Proof" of 3. Suppose A and B are invertible $n \times n$ matrices. We can think of A as a functions that takes a vector $\mathbf{x}$ from $\mathbb{R}^n$ and gives as output a vector $\mathbf{y} = A\mathbf{x}$ in $\mathbb{R}^n$. Likewise for B and the product AB. Denote these functions by f_A, f_B and f_{AB} respectively. Then $f_{AB} = f_A \circ f_B$. All three functions are invertible, so $f_{AB}^{-1} = f_B^{-1} \circ f_A^{-1}$.

If you still do not see this, consider this example. Suppose I asked what value of x makes $5 = \sqrt{\ln x}$? *First* you would find $25 = 5^2$, *second* you would find e^{25}. That is, you'd do the inverse of each operation is *reverse* order! □

Proof of 4. Suppose A is invertible. We need to solve $A^{-1}X = I$. But by part 2, $X = A$ is the solution.

Proof of 5. Suppose A is invertible. We start with the fact the $A^{-1}A = I$. Transpose both sides to get $(A^{-1}A)^T = I^T = I$. We then get $A^T(A^{-1})^T = I$. Thus $X = (A^{-1})^T$ is the solution to $A^T X = I$, and so we are done.

Problem 3. Find two invertible matrices such that $A + B$ is invertible but $(A + B)^{-1} \neq A^{-1} + B^{-1}$.

Problem 4. Prove that if A is any $n \times n$ matrix such that one column of A is a multiple of another column of A then A is not invertible.

Problem 5. Prove that $(cA)^{-1} = \frac{1}{c}A^{-1}$ for A invertible and $c \neq 0$.

Problem 6. Let $k \geq 1$. Let $A = A_1 A_2 \cdots A_k$, where each A_i is invertible. Use the Principle of Mathematical Induction to prove that $A^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}$.

Problem 7. Assuming the matrices A, B, and C are invertible and that the sums and products are defined, prove or disprove the following proposed identities.

(a) $(AB^T)^{-1} = ((BA^T)^{-1})^T$

(b) $A^{-1}(B - C)^T = ((B^T - C^T)^{-1}A)^{-1}$.

(c) $(5A^T B)^{-1} = \frac{1}{5}B^{-1}(A^{-1})^T$.

(d) $(3A^{-1} + B)^T A^T = 3I + (AB)^T$.