

# Lines and Planes<sup>1</sup>

## 1 Lines in the Plane

Every line of points  $L$  in  $\mathbb{R}^2$  can be expressed as the solution set for an equation of the form  $Ax + By = C$ . Will we call this the *ABC form*. Recall that the *slope-intercept* form is  $y = ax + b$ . However, vertical lines cannot be expressed in slope-intercept form, but they can be expressed in ABC form: The vertical line  $x = 5$  corresponds to,  $A = 1$ ,  $B = 0$ , and  $C = 5$ .

The ABC form for a line  $L$  is not unique for if we multiply both sides by any nonzero number the solution set is unchanged. Any line  $L$  can also be expressed by a pair of *parametric equations*, by which we mean:

$$\begin{aligned}x(t) &= at + b \\y(t) &= ct + d\end{aligned}$$

for suitable constants. These can be rewritten in *vector form* as  $\langle x, y \rangle = \langle a, c \rangle t + \langle b, d \rangle$ . The vectors  $\langle a, c \rangle$  and  $\langle b, d \rangle$  have a nice geometric/physical interpretation.

Regard  $t$  as time. One can imagine a particle moving along  $L$  in accordance with the given parametric equations. We let  $\mathbf{p}(t) = \langle x(t), y(t) \rangle$  and call it the *position vector*. Then  $\mathbf{p}(0) = \langle b, d \rangle$  is the *initial position*. Notice,

$$\frac{d\mathbf{p}}{dt} = \langle x'(t), y'(t) \rangle = \langle a, c \rangle$$

(The derivative of a vector of functions is just the vector given by taking the derivative of each component, but we will not need this in this course.) Thus, we call  $\mathbf{v} = \langle a, c \rangle$  the *velocity vector*. It is parallel to  $L$ . It is customary to place its base point on  $L$ . See Figure 1(left side).

We now give a geometric interpretation for the ABC form of an equation of a line. First, suppose  $C = 0$ ; this just means the line  $L$  goes through the origin. Let  $\mathbf{n} = \langle A, B \rangle$ , and again set  $\mathbf{p} = \langle x, y \rangle$ . Then we have  $\mathbf{n} \bullet \mathbf{p} = 0$ . That is the vectors  $\mathbf{n}$  and  $\mathbf{p}$  are at right angles to each other. Thus, the line  $L$  for  $Ax + By = 0$  is the set of all points  $(x, y)$  such that  $\langle x, y \rangle$  is perpendicular to  $\langle A, B \rangle$ .

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Now we consider the general case:  $Ax + By = C$ . Pick some particular point on the line and call it  $(x_0, y_0)$ . Then  $C = Ax_0 + By_0$ . Therefore, for any point  $(x, y)$  on  $L$  we have  $Ax + By = Ax_0 + By_0$ . We can rewrite this as

$$\begin{aligned} Ax - Ax_0 + By - By_0 &= 0 \\ A(x - x_0) + B(y - y_0) &= 0 \\ \langle A, B \rangle \bullet \langle x - x_0, y - y_0 \rangle &= 0 \\ \mathbf{n} \bullet (\langle x, y \rangle - \langle x_0, y_0 \rangle) &= 0 \\ \mathbf{n} \bullet (\mathbf{p} - \mathbf{p}_0) &= 0 \end{aligned}$$

In the last line we have let  $\mathbf{p}_0 = \langle x_0, y_0 \rangle$ . The vector  $\mathbf{p} - \mathbf{p}_0$  can be thought of as lying in  $L$  with its tail at  $(x_0, y_0)$  and its head at  $(x, y)$ .

Thus,  $L$  is the unique line perpendicular to the vector  $\mathbf{n} = \langle A, B \rangle$  that passes through  $(x_0, y_0)$ . See Figure 1(right side). The vector  $\mathbf{n}$  is called a *normal vector* for the line  $L$ . Given a vector to use as normal vector and a point we can easily find an equation for the corresponding line.

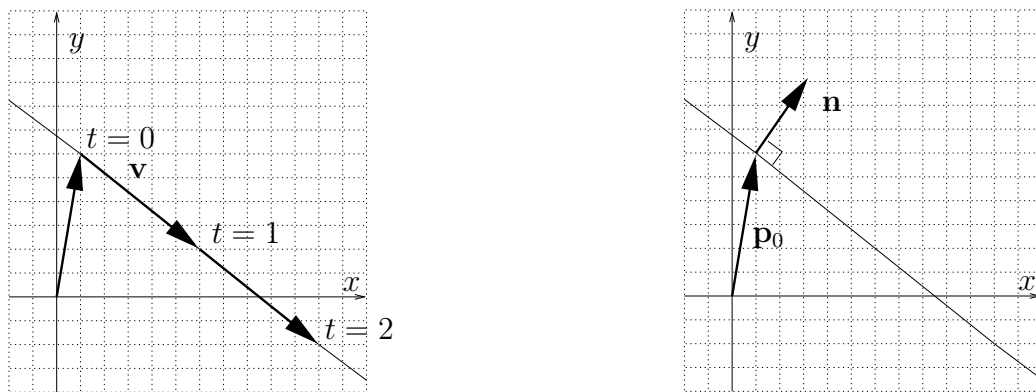


Figure 1: Left: A parametric line. Right: Normal vector to a line.

**Problem 1.** Consider the line that is determined by the parametric equations  $x(t) = 3t - 2$  and  $y(t) = -t + 7$ . Find an equation for the line in ABC form.

**Problem 2.** Consider the line determined by  $4x - 7y = 2$ . Find a pair of parametric equations for this line.

Note: Problems 1 and 2 have many correct answers.

## 2 Lines and Planes in 3-space

The three dimensional set  $\mathbb{R}^3$  is the set of all triples  $(x, y, z)$  where  $x$ ,  $y$ , and  $z$  are real numbers. Such a triple is called the *xyz*-coordinates of a point. These are also called *rectilinear coordinates*. The set  $\{(x, 0, 0) \mid x \in \mathbb{R}\}$  is the  $x$ -axis. The  $y$  and  $z$  axes are defined similarly. They are clearly lines. The set  $\{(x, y, 0) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$  is the  $xy$ -plane. The  $yz$  and  $xz$  planes are defined similarly. Visualizing structures in three dimensions takes practice.

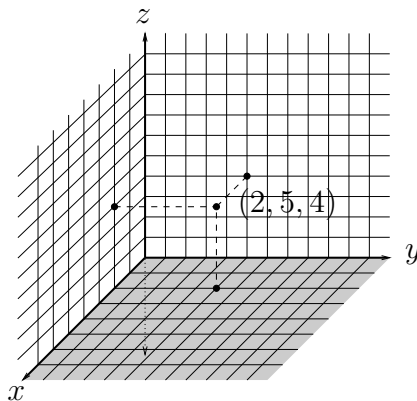


Figure 2: Three dimensional space:  $\mathbb{R}^3$

Any line  $L$  in  $\mathbb{R}^3$  can be expressed parametrically in the form:

$$\begin{aligned}x(t) &= at + b \\y(t) &= ct + d \\z(t) &= et + f\end{aligned}$$

or, in vector form,  $\langle x, y, z \rangle = \langle a, c, e \rangle t + \langle b, d, f \rangle$ . As with lines in  $\mathbb{R}^2$  it is useful to think of  $\langle a, c, e \rangle$  as a velocity vector and  $\langle b, d, f \rangle$  as the position at  $t = 0$ .

However, there is no way to express a line in  $\mathbb{R}^3$  as a single equation in the three variables.

Now we move on to planes in  $\mathbb{R}^3$ . Consider the solution set of an equation of the form  $Ax + By + Cz = D$ . If  $A = B = C = D = 0$ , the solution set is all of  $\mathbb{R}^3$ . If  $A = B = C = 0$  but  $D \neq 0$  the solution set is empty. These two examples should be thought of as degenerate cases. In all other cases the solution set to an equation of the form  $Ax + By + Cz = D$  will be a plane in

$\mathbb{R}^3$ . We will call this form the *ABCD form*. Parametric equation for planes are described in the next section.

**Example 1.** Convince yourself of the following:

- If  $A = B = D = 0$  and  $C \neq 0$  then  $Ax + By + Cz = D$  is the  $xy$ -plane.
- If  $A = C = D = 0$  and  $B \neq 0$  then  $Ax + By + Cz = D$  is the  $xz$ -plane.
- If  $B = C = D = 0$  and  $A \neq 0$  then  $Ax + By + Cz = D$  is the  $yz$ -plane.

**Example 2.** Graph the plane  $P$  that is the solution set to  $2x + 3y + 4z = 12$  in  $\mathbb{R}^3$ .

*Solution.* Will we find the intercepts with each of the three coordinate axes. Let  $y = z = 0$ . Then  $x = 6$ . Hence the point  $(6, 0, 0)$  is on  $P$ . Next let  $x = z = 0$ . Then  $y = 4$ . Hence the point  $(0, 4, 0)$  is on  $P$ . Finally let  $x = y = 0$ . Then  $z = 3$ . Hence  $(0, 0, 3)$  is on  $P$ . In Figure 3 we plot these points and connect them with line segments to help visualize the plane  $P$ .  $\square$

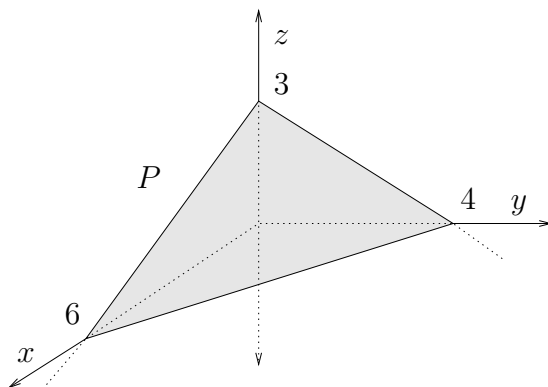


Figure 3: The plane given by  $2x + 3y + 4z = 12$  in  $\mathbb{R}^3$ .

Next we will give a geometric interpretation of an equation for a plane  $P$  in ABCD form. First we consider the case where  $D = 0$ . Let  $\mathbf{n} = \langle A, B, C \rangle$  and  $\mathbf{p} = \langle x, y, z \rangle$ . Then the equation  $Ax + By + Cz = 0$  becomes  $\mathbf{n} \bullet \mathbf{p} = 0$ . Thus, the solution set is the plane  $P$ , passing through the origin of  $\mathbb{R}^3$  whose points, when regarded as vectors, are perpendicular to  $\mathbf{n}$ .

We return to the general case:  $Ax + By + Cz = D$ . Let  $\mathbf{p}_0 = \langle x_0, y_0, z_0 \rangle$  be some fixed point that satisfies the given equation. We leave it to the reader to show that

$$\mathbf{n} \bullet (\mathbf{p} - \mathbf{p}_0) = 0.$$

Thus, the solution set of  $Ax + By + Cz = D$  is the unique plane passing through  $\mathbf{p}_0$  and perpendicular to  $\mathbf{n} = \langle A, B, C \rangle$ .

**Example 3.** We reconsider the plane  $P$  given by  $2x + 3y + 4z = 12$  in Example 2. Let  $\mathbf{n} = \langle 2, 3, 4 \rangle$ . Pick two points on  $P$ , say  $\mathbf{p}_1 = (2, 2, 1/2)$  and  $\mathbf{p}_2 = (4, 0, 1)$ . Regard them as vectors and let  $\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1 = \langle 2, -2, 1/2 \rangle$ . Then  $\mathbf{v} \bullet \mathbf{n} = 2 \cdot 2 + (-2) \cdot 3 + (1/2) \cdot 4 = 4 - 6 + 2 = 0$  as expected.

**Example 4.** Let  $P_1$  be the plane given by  $2x + 3y - z = 5$  and let  $P_2$  be the plane given by  $x + y + z = 1$ . Find parametric equations for the line  $L = P_1 \cap P_2$ , then rewrite them in vector form.

*Solution.*

$$\left. \begin{array}{l} 2x + 3y - z = 4 \\ x + y + z = 1 \end{array} \right\} \implies y - 3z = 2.$$

Let  $z = t$ . Then  $y = 3t + 2$  and  $x = 1 - y - z = -4t - 1$ . Thus,

$$\begin{aligned} x(t) &= -4t - 1 \\ y(t) &= 3t + 2 \\ z(t) &= t \end{aligned}$$

are parametric equations for the line  $L$ . Lastly, we can rewrite these in vector form.  $\langle x, y, z \rangle = \langle -4, 3, 1 \rangle t + \langle -1, 2, 0 \rangle$ .  $\square$

**Example 5.** Find an equation for the plane passing through the three points  $(1, 1, 1)$ ,  $(1, 2, 3)$ , and  $(2, -1, 0)$ .

*Solution.* We have three conditions and these give us three equations in four

unknowns.

$$\left. \begin{array}{rcl} A + B + C & = & D \\ A + 2B + 3C & = & D \\ 2A - B & = & D \end{array} \right\} \implies$$

$$\left. \begin{array}{rcl} A + B + C & = & D \\ B + 2C & = & 0 \\ -3B - 2C & = & -D \end{array} \right\} \implies$$

$$\left. \begin{array}{rcl} A + B + C & = & D \\ B + 2C & = & 0 \\ C & = & -D/4 \end{array} \right\} \implies$$

$$\left. \begin{array}{rcl} A + B & = & 5D/4 \\ B & = & D/2 \\ C & = & -D/4 \end{array} \right\} \implies \begin{array}{l} A = 3D/4 \\ B = D/2 \\ C = -D/4 \end{array}$$

Any nonzero value of  $D$  will do. Let  $D = 4$ . Then  $3x + 2y - z = 4$  is an equation for our plane.  $\square$

**Problem 1.** Consider the three points  $(1, 1, 1)$ ,  $(2, 0, 2)$ , and  $(4, -2, 4)$ . Show that they do not determine a unique plane because they lie on the same line. Find an equation for this line; write it in vector form.

**Problem 2.** Let  $P$  be the plane given by  $x + 2y - 3z = 1$ . Let  $L_{xy}$  be the intersection of  $P$  with the  $xy$ -plane. Find an equation for this line ABC form and slope-intercept form.

**Problem 3.** Graph, separately, each of the planes determined by these three equations:  $2x + 2y - 3z = 1$ ,  $x + 2y + 4z = -1$ , and  $3x - 2y - 2z = 7$ .

**Problem 4.** Find the point of intersection of the three planes determined by these three equations:  $2x + 2y - 3z = 1$ ,  $x + 2y + 4z = -1$ , and  $3x - 2y - 2z = 7$ .

**Problem 5.** Show that the two planes determined by  $2x + 2y - 3z = 1$  and  $4x + 4y - 6z = 0$  do not intersect and are thus parallel.

**Problem 6.** Let  $P$  be the plane given by  $2x + 3y - 2z = 1$ . Let  $L$  be the line given by  $\langle x, y, z \rangle = \langle 1, 1, 1 \rangle t + \langle 1, 0, 1 \rangle$ . Find the point where they meet.

**Problem 7.** Show that these four points lie in the same plane:  $(1, 1, -1)$ ,  $(-1, 0, 0)$ ,  $(-1, 1, -\frac{1}{2})$ , and  $(1, -1, 0)$ . Find an equation for this plane.

### 3 Parametric Equation for a Plane

There is another form for equations of planes in  $\mathbb{R}^3$  that is the analog of the parametric form for equations of a line. The difference is we will need two parameters,  $r$  and  $s$ , instead of one. Of course, the time metaphor is no longer useful.

Let  $P$  be a plane given by  $Ax + By + Cz = D$ . Assume that  $C \neq 0$ . Then we can solve for  $z$  and get  $z = D/C - A/Cx - B/Cy$ . (If  $C = 0$  solve for  $x$  or  $y$  instead.) Think of  $z$  as the height above the  $xy$ -plane. Now let  $x = r$  and  $y = s$ , and think of  $r$  and  $s$  as free parameters. We can now write

$$\begin{aligned}\langle x, y, z \rangle &= \langle r, s, D/C - A/Cr - B/Cs \rangle \\ &= \langle 0, 0, D/C \rangle + r \langle 1, 0, -A/C \rangle + s \langle 0, 1, -B/C \rangle\end{aligned}$$

This equation is far from unique. We can start with any point  $(x_0, y_0, z_0) \in P$ , regard it as a vector  $\mathbf{p}_0 = \langle x_0, y_0, z_0 \rangle$  and add multiples of  $\langle 1, 0, -A/C \rangle$  and  $\langle 0, 1, -B/C \rangle$  to it and stay in the plane. Furthermore, if we let  $\mathbf{v}_1$  and  $\mathbf{v}_2$  be nonzero multiples of  $\langle 1, 0, -A/C \rangle$  and  $\langle 0, 1, -B/C \rangle$ , respectively then

$$\mathbf{p} = \mathbf{p}_0 + r\mathbf{v}_1 + s\mathbf{v}_2$$

gives the same plane  $P$ . Indeed, we could use any pair of vectors in  $P$  with tails at  $\mathbf{p}_0$  as long as they point in different directions.

We will use this formulation to place a coordinate system on  $P$ . Take a point  $(x_0, y_0, z_0)$  on  $P$  and call it the origin of  $P$ . Then any point on  $P$  can be gotten to by adding multiples of  $\mathbf{v}_1$  and  $\mathbf{v}_2$  to  $\mathbf{p}_0$ . Thus, for any point on  $P$  we can think of it as having coordinates  $(r, s)$ . See Figure 4.

**Example 1.** Define a plane  $P$  by

$$\langle x, y, z \rangle = \langle 1, 2, 3 \rangle + r \langle 1, 1, 0 \rangle + s \langle 0, 1, 1 \rangle$$

Show that the point  $(0, 2, 4)$  is on  $P$  and find its  $rs$ -coordinates.

*Solution.* We have three equations and two unknowns.

$$\left. \begin{array}{l} 0 = 1 + 1r + 0s \\ 2 = 2 + 1r + 1s \\ 4 = 3 + 0r + 1s \end{array} \right\} \implies \begin{array}{l} r = -1 \\ s = 1 \end{array}$$

Thus,  $(0, 2, 4) \in P$  and it has  $rs$ -coordinates  $(-1, 1)$  relative to the given parametric equation.  $\square$

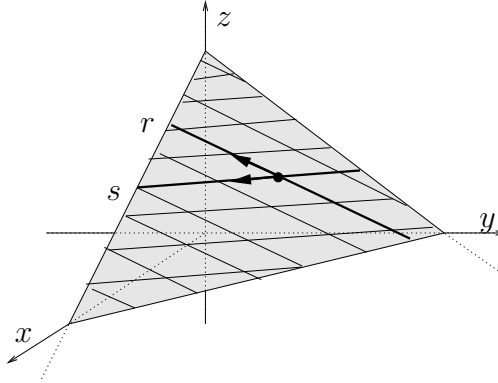


Figure 4: Coordinates for a plane: The dark lines are the  $r$  and  $s$ -axes

**Problem 1.** Using the same plane  $P$  in Example 1, find the  $rs$ -coordinates of  $(3, 3, 2)$ .

**Problem 2.** Show that the point  $(1, 2, -1)$  is not on the plane  $P$  of Example 1.

**Problem 3 (Hard).** The equation  $2r + 3s = 1$  determines a line  $L$  in the plane  $P$  of Example 1, using  $rs$ -coordinates. Find a parametric equation for  $L$  in  $xyz$ -coordinates.

## 4 Summary

**I.** For lines in  $\mathbb{R}^2$  we have studied three forms, the **ABC form**, **parametric form**, and **vector form**.

$$Ax + By = C \quad \begin{array}{l} x(t) = at + b \\ y(t) = ct + d \end{array} \quad \langle x, y \rangle = \langle a, c \rangle t + \langle b, d \rangle$$

**II.** For lines in  $\mathbb{R}^3$  we do not have an analog of the ABC form, but we can express any line in  $\mathbb{R}^3$  in **parametric** and **vector form**.

$$\begin{array}{l} x(t) = at + b \\ y(t) = ct + d \\ z(t) = et + f \end{array} \quad \langle x, y, z \rangle = \langle a, c, e \rangle t + \langle b, d, f \rangle$$

**III.** For planes in  $\mathbb{R}^3$  we studied the **ABCD form**, the **parametric form**, and the **vector form**.

$$\begin{array}{rcl} & x(r, s) & = ar + bs + c \\ Ax + By + Cz = D & y(r, s) & = dr + es + f \\ & z(r, s) & = gr + hs + i \end{array}$$

$$\langle x, y, z \rangle = \langle a, d, g \rangle t + \langle b, e, f \rangle + \langle c, f, i \rangle$$

For all three cases the you should be able to convert equations of one form into the others. You should begin to wonder what sorts of linear structures exist in  $\mathbb{R}^n$  for  $n > 3$ .