Row, Column and Null Spaces [1]

We are now in a position to prove the claim made in the Solving Linear Systems handout that "two systems of linear equations have the same solutions sets if and only if the associated augmented matrices are row equivalent."

Definition .1. Let A be an $m \times n$ matrix. The **column space** of A is the span of its column vectors. The **row space** of A is the span of its row vectors. The solution set of the homogeneous problem, $A\mathbf{v} = \mathbf{0}$, is the **null space** of A. The dimension of the Null Space of A is sometimes called the **nullity** of A.

Earlier we showed that the null space of a matrix is a vector space.

Example 1. Let

$$A = \begin{bmatrix} 0 & 3 & 2 & 3 & 2 & 4 & 7 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 3 & 2 & 9 & 9 & 1 & 0 \\ 1 & 3 & 3 & 4 & 2 & 6 & 8 & 2 & 0 \end{bmatrix}$$

Find bases for the row space, column space and null space of A.

Solution. We put A into reduced row echelon form.

$$\text{rref}\,(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & -2/3 & 0 & -1 & 5/3 & 1 \\ 0 & 1 & 2/3 & 0 & 0 & 0 & 1 & 5/3 & 5/3 \\ 0 & 0 & 0 & 1 & 2/3 & 0 & 0 & -8/3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Space. First let's think about the row space. The rows of A and rref$\,(A)$ are linear combinations of each other and so span the same space. But, it is easy to see that the nonzero rows of rref$\,(A)$ are linearly independent. This is because of the pivots. Row 2 in rref$\,(A)$ cannot be a multiple of row 1, and row 3 cannot be a linear combination of rows 1 and 2, and row 4 cannot be a linear combination of the first three rows. It is an exercise (problem 2 below) to show that this is sufficient to establish linear independence. Thus, the nonzero rows of rref (A) are a basis for the row space of A.

Column Space. The nonzero rows of rref (A^T) give a basis for the column space of A. Below we give another way to find a basis of the column space of A that does not require us to calculate rref (A^T) if we already have rref (A).

The pivot column vectors of rref (A) are certainly linearly independent and so form a basis of the column space of rref (A), but they do not span the column space of A. For example, the column space of A contains the first column of A, but any linear combination of the pivot columns of rref (A) will have a zero for its last entry.

It turns out however, that the columns of A that correspond to pivot columns of rref (A), that is columns 1, 2, 4 and 6, form a basis for the column space of A. Here is how to see this. In rref (A) column 3 is equal to column 1 plus 2/3 times column 2. This same relationship holds for columns 1, 2, and 3 of A. This is no coincidence. Row operations do not effect relations among the columns. More precisely, suppose $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k$ are column vectors such that $\mathbf{v}_0 = \Sigma_{i=1}^{k} c_i \mathbf{v}_i$. Form a matrix $[\mathbf{v}_0 \mathbf{v}_1 \cdots \mathbf{v}_k]$ and apply any of the row operations to it. Let the resulting matrix be $[\mathbf{w}_0 \mathbf{w}_1 \cdots \mathbf{w}_k]$. Then, it is an easy exercise (problem 1 below) to show that $\mathbf{w}_0 = \Sigma_{i=1}^{k} c_i \mathbf{w}_i$, where the c_i's are the same as before.

Since, any column of rref (A) is a linear combination of the pivot column of rref (A) it follows any column of A is linear combinations of the columns vectors of A that correspond to the pivots. Hence columns 1, 2, 4 and 6, of A span the row space of A. Since no nontrivial linear combination of the pivot columns of rref (A) is the zero vector, the same is true for the corresponding columns of A. (You can check these claims explicitly for this example.) Hence, columns 1, 2, 4 and 6 of A form a basis for the column space of A.

Null Space. Now for the null space. We know how to find a set of vectors that span the solution space of the homogeneous problem $A\mathbf{x} = \mathbf{0}$. Let's use $a_1, a_2, \ldots, a_9$ as the variable names. Then a_1, a_2, a_4 and a_6 will be our dependent variables. Solving

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -2/3 & 0 & -1 & 5/3 & 1 \\ 0 & 1 & 2/3 & 0 & 0 & 0 & 1 & 5/3 & 5/3 \\ 0 & 0 & 0 & 1 & 2/3 & 0 & 0 & -8/3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

gives

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix} = a_3 \begin{bmatrix} -1 \\ -2/3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_5 \begin{bmatrix} 2/3 \\ 0 \\ 0 \\ -2/3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_7 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + a_8 \begin{bmatrix} -5/3 \\ -5/3 \\ 0 \\ 8/3 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a_9 \begin{bmatrix} -1 \\ -5/3 \\ 0 \\ 3 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The five vectors on the right-hand side span the null space. They are linearly independent because for entries 3, 5, 7, 8 and 9 one and only one of the five vectors has a nonzero entry. Hence none can be a linear combination of the others. Thus we have a basis for the null space.

Summary. The methods used in Example 1 work in general. For any $m \times n$ matrix A we can find bases for the row, column and null spaces by computing $\text{rref}(A)$. Then

- the rowes of $\text{rref}(A)$ are a basis for the row space,
- the columns of A corresponding to the pivots of $\text{rref}(A)$ are a basis for the column space, and
- a basis for the null space can be found by solving $\text{rref}(A)\mathbf{x} = \mathbf{0}$.

Armed with our understanding of vector spaces we can prove the following theorems.

Theorem .2. *Let A be an $m \times n$ matrix. The dimensions of the row and column spaces of A are both equal to the rank of the matrix. The dimension of the null space plus the rank of A is equal to n.*

Proof. The first claim is just the observation that dimension is well defined and the definition of rank. The second is just noting that the dimension of the null space is well defined and is the number of columns minus the numbet of pivots. □

Remark. While the dimension of the row and column space of a matrix are equal the spaces themselves are not generally the same.

Theorem .3. *Let A and B be $m \times n$ matrices. Then $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution spaces if and only if $rref(A) = rref(B)$.*

Theorem .4. *Let A be an $m \times n$ matrix. Then A is row equivalent to one and only one matrix in reduced row echelon form.*

Analogy. Suppose we want to tell if two fractions, say 12/30 and 14/35, are equal as rational numbers – they are different *symbols*. We would cancel out any common divisors in each, putting them into reduced form. They are equal as numbers if and only if their respective reduced forms are identical. In our example 12/30 and 14/35 both reduce to 2/5 and thus represent the same number. This is analogous to what we have done with matrices.

Problem 1. Prove that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent if and only if for each $j = 2, .., k$ $\mathbf{v}_j$ is not a linear combination of the vectors with smaller index.

Problem 2. For the matrix $[\mathbf{v}_0 \cdots \mathbf{v}_k]$ apply any of the row operations to it. Let the resulting matrix be $[\mathbf{w}_0 \cdots \mathbf{w}_k]$. Show that if $\mathbf{v}_0 = \Sigma_{i=1}^k c_i \mathbf{v}_i$ then $\mathbf{w}_0 = \Sigma_{i=1}^k c_i \mathbf{w}_i$.

Problem 3. Find bases for the row space, column space and null space for each matrix below. Check that the nullity plus the rank equals the number of columns.

a. $\begin{bmatrix} 2 & 0 & -4 & -6 \\ -1 & 0 & 2 & 3 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 4 & -2 \\ 1 & 2 & 0 \\ -3 & -6 & 2 \end{bmatrix}$ c. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 3 \\ 2 & 2 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 3 & 3 & 3 & 0 & -6 \\ 0 & 0 & 2 & 6 & -4 \\ 1 & 1 & 3 & 7 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$

Affine Spaces: Optional Reading

Definition .5. Let V be a vector space. Let W be a subspace and let $\mathbf{v}$ be a nonzero vector in V. Then we let $W + \mathbf{v}$ denote the set $\{\mathbf{w} + \mathbf{v} \,|\, \mathbf{w} \in W\}$. A subset of V that is not a vector space but that can be expressed in this form is called an **affine space**. In this case the vector $\mathbf{v}$ is called an **offset vector**.

Example 2. The solution set of $\begin{bmatrix} 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [2]$ can be written as the set of all vectors of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} t$$

where t can be any real number. It is thus an affine space. (Of course it is just the line $x + 2y = 1$, which does not go through the origin.)

Problem 4. Let $\mathbf{b}$ be a nonzero column vector and A be a matrix of an appropriate size. Prove that the solution set of $A\mathbf{x} = \mathbf{b}$ is an affine space if it is nonempty.

The next two theorems are easy to prove.

Theorem .6. *Let V be a vector space. Let $\mathbf{v}$ and $\mathbf{v}'$ be nonzero vectors in V. Let W and W' be subspace of V. If $W + \mathbf{v} = W' + \mathbf{v}'$ then $\dim W = \dim W'$.*

Definition .7. Let A be an affine space in a vector space V. Suppose $A = \mathbf{v} + W$ for a vector subspace $W \subset V$ and a nonzero $\mathbf{v} \in V$. The we define the dimension of A to be the dimension of W.

Theorem .8. *Two systems $A\mathbf{x} = \mathbf{b}$ and $B\mathbf{x} = \mathbf{d}$ have the same solution sets if and only if $rref([A|\mathbf{b}]) = rref([B|\mathbf{d}])$.*