Sums of Powers [1]

In the Induction Handout we proved the summation formula

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}. \qquad (1)$$

You might have wondered, how was the formula discovered in the first place? Figure 1 shows how this might have been done. The sum in question can be represented as the area under the "staircase" in the figure. The area under the diagonal is $n^2/2$ while the area below the stairs but above the diagonal is just half the number of steps or $n/2$. Thus the total area is $(n^2+n)/2$, which is equivalent to formula (1).

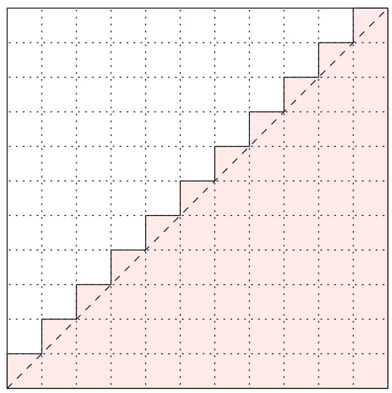

Figure 1:

Next we consider the sum of the first n squares: $\sum_{k=1}^{n} k^2$. One can discover a formula for this sum by imagining a stack of unit cubes. The first layer is a square formed by $n \times n$ unit cubes. On top of it is a layer of $(n-1)\times(n-1)$ unit cubes, and so on. On the very top of the stack is a single cube. The total volume is $\sum_{k=1}^{n} k^2$. A formula for this volume can be derived by using basic geometry. Do this. The formula is $\frac{n(n+1)(2n+1)}{6}$. Its proof was a problem in the Induction Handout (Why prove it? Why not just trust the geometry?)

What about sums of higher powers? Geometry is no longer helpful. But we notice a pattern. The sum of the first n numbers is a quadratic polynomial. The sum of the first n squares is a cubic polynomial. Perhaps the sum of the first n cubes is a quartic polynomial. We will use linear algebra to find a candidate. Suppose $p(x) = ax^4 + bx^3 + cx^2 + dx + e$ is a fourth degree polynomial such that

$$\sum_{k=1}^{n} k^3 = p(n).$$

By direct computation $p(1) = 1$, $p(2) = 9$, $p(3) = 36$, $p(4) = 100$, and $p(5) = 225$. This the gives 5 equations in 5 unknowns show below.

$$\begin{aligned}
a + b + c + d + e &= 1 \\
16a + 8b + 4c + 2d + e &= 9 \\
81a + 27b + 9c + 3d + e &= 36 \\
256a + 64b + 16c + 4d + e &= 100 \\
625a + 125b + 25c + 5d + e &= 225
\end{aligned}$$

We shall solve this system with the aid of Maple.

```
>  Y:=Matrix([[1],[9],[36],[100],[225]]):
>  A := Matrix([
>  [1,     1,  1, 1, 1],
>  [16,    8,  4, 2, 1],
>  [81,   27,  9, 3, 1],
>  [256,  64, 16, 4, 1],
>  [625, 125, 25, 5, 1]]):
>  LinearSolve(A,Y);
```

$$\begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \\ 0 \\ 0 \end{bmatrix}$$

Thus $p(n) = 1/4n^4 + 1/2n^3 + 1/4n^2$. But how do we *know* this formula will work for all n? Even if there was no polynomial solution to our original problem, Maple would have found this polynomial. We shall check it with induction, using Maple to do the algebra. First we define $p(n)$ as a function in Maple.

```
> p:=n -> 1/4*n^4+1/2*n^3+1/4*n^2;
```

Clearly this polynomial formula works for $n = 1$. Assume it holds for some fixed value of n. We must show it works for $n+1$. That is, we need to show that $p(n+1) = \sum_{k=1}^{n+1} k^3$. But $\sum_{k=1}^{n+1} k^3$ is equal to $\sum_{k=1}^{n} k^3 + (n+1)^3$ which equals $p(n)+(n+1)^3$. So, all we have to do to check that $p(n+1) = p(n)+(n+1)^3$. We do this with Maple:

```
> expand(p(n+1));
```

$$\frac{1}{4}n^4 + \frac{3}{2}n^3 + \frac{13}{4}n^2 + 3\,n + 1$$

```
> expand(p(n) + (n+1)^3);
```

$$\frac{1}{4}n^4 + \frac{3}{2}n^3 + \frac{13}{4}n^2 + 3\,n + 1$$

Thus, by the Principle of Mathematical Induction we have proved that

$$\sum_{k=1}^{n} k^3 = 1/4n^4 + 1/2n^3 + 1/4n^2.$$

Problem 1. Find and prove formulas for $\sum_{k=1}^{n} k^p$ for (a) $p = 4$, (b) $p = 5$, and (c) $p = 6$.