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Vectors[1]

0.1 Introduction

On one level a vector is just a point; we can regard every point in $\mathbb{R}^2$ as a vector. When we do so we will write $\langle a, b\rangle$ instead of the usual (a, b). In most physical problems vectors are regarded as directed magnitudes; they have a size and a direction. To emphasize this vectors are drawn as arrows in the plane. Often the base of the arrow is at the origin and the head is at the point (a, b) for the vector $\langle a, b\rangle$. In many applications the base of the arrow is placed elsewhere.

Suppose you are lost in a dense forest and you finally make radio contact with your base camp. They pick up your GPS single and tell you to "go North 3 miles". This is a vector; we have a direction, North, and a magnitude, 3 miles. In the xy-plane we could represent it by $\langle 0, 3\rangle$. You of course would assume that the starting point was your current location. (We take the positive y-axis to be North, the positive x-axis to be East, and so on.)

Suppose the weather station says the wind is blowing 10 miles per hour from the NE. The vector would be 10 units long and pointing SW. We could wright this as $\left\langle -\sqrt{50}, -\sqrt{50}\right\rangle$. (You should check that the length is 10.) Its location is understood to be everywhere in the region the weather reporter was referring to.

The velocity vector of a moving particle might be placed with its arrow based at the location of the particle. Thus, if $x(t) = t^2$ and $y(t) = t^3$ describes the motion of a particle in the xy-plane, then the velocity vector at $t = 2$ is $\langle x'(2), y'(2)\rangle = \langle 4, 12\rangle$. It would be natural to place its base at $(4, 8)$ and its arrow tip at $(4+4, 8+4)$, i.e. $(8, 12)$. The velocity vector in and of itself has no location. This could of course be applied to a parametrized curve in $\mathbb{R}^3$.

Consider a ball in $\mathbb{R}^3$ spinning about a line that passes through its center at a rate of R revolutions per second. Its *spin vector* would be parallel to the line of rotation and its magnitude would R. It direction along the lien of rotation is determined by the right-hand rule: wrap your right hand around the ball with your thumb parallel to the line and the other four fingers are pointing in the direction of the rotation. Your thump will, by convention, be pointing in the direction of the spin vector. For example if the ball was centered at (0,0,0) and spinning at 3 revolutions per second about the z-

axis counterclockwise when looked at from above, its spin vector would be $\langle 0,0,3\rangle$. One would normally think of the spin vector as having its base at the center of the ball, but the spin vector itself is just a direction and a magnitude. If we moved the ball's center to (4,2,18) without changing its spin, its spin vector is still $\langle 0,0,3\rangle$.

0.2 Definitions and Properties

The difference between points and vectors is that while points are purely geometric, the set of vectors in the plane is endowed with an algebraic structure. We define the following three binary operations, first for vectors in $\mathbb{R}^2$:

- Vector addition: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\langle a_1, b_1\rangle + \langle a_2, b_2\rangle = \langle a_1 + a_2, b_1 + b_2\rangle$.
- Scalar multiplication: $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is given by $r\,\langle a, b\rangle = \langle ra, rb\rangle$.
- The dot product: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is given by $\langle a_1, b_1\rangle \bullet \langle a_2, b_2\rangle = a_1a_2 + b_1b_2$.

Vectors and these vector operations can be defined for any $\mathbb{R}^n$. Let $\mathbf{v} = \langle v_1, v_2, \ldots, v_n\rangle$ and $\mathbf{u} = \langle u_1, u_2, \ldots, u_n\rangle$ be vectors in $\mathbb{R}^n$. Then:

- $\mathbf{v} + \mathbf{u} = \langle v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n\rangle$.
- For $r \in \mathbb{R}$, $r\mathbf{v} = \langle rv_1, rv_2, \ldots, rv_n\rangle$.
- $\mathbf{v} \bullet \mathbf{u} = v_1u_1 + v_2u_2 + \cdots + v_nu_n = \displaystyle\sum_{i=1}^{n} v_iu_i$.

Example 1 (Sample Calculations).

1. $\langle 2,3,5\rangle + \langle 3,0,-1\rangle = \langle 5,3,4\rangle$.
2. $14\,\langle 1/7, 1/2, -1, 2, 0\rangle = \langle 2, 7, -14, 28, 0\rangle$.
3. $\langle -3, 1/2, 2, 0\rangle \bullet \langle 1,2,3,4\rangle = -3 + 1 + 6 + 0 = 5$.
4. $\langle 5,3,4\rangle \bullet \langle 1,1,1,1\rangle$ is undefined.
5. $(\langle 3,7,1\rangle + \langle 2,-3,4\rangle) \bullet \langle 2,-1,6\rangle = \langle 7,4,5\rangle \bullet \langle 2,-1,6\rangle = 16$
6. If $\langle 2,1,4,5\rangle + \mathbf{v} = \langle 5,3,4,1\rangle$, then $\mathbf{v} = \langle 3,2,0,-4\rangle$

Problem 1. Perform the following calculations if possible.

(a) $\langle 2,0,2,2\rangle \bullet \langle 0,2,1,-1\rangle$
(b) $\langle 2,3\rangle + \langle 7,8\rangle \bullet \langle 3,5\rangle$
(c) $3\,\langle 7,-3\rangle + 2\,\langle 4,3\rangle$
(d) $12(\langle 1,-2,3\rangle - \langle 2,3,6\rangle)$
(e) $7\,(\langle 2,3\rangle \bullet \langle 2,3\rangle)$
(f) $\sqrt{\langle 3,3,4,0,1\rangle \bullet \langle 3,3,4,0,1\rangle}$

Definition 1. We will use a bold face zero $\mathbf{0}$ to denote the vector of all zero's for any $\mathbb{R}^n$. That is $\mathbf{0} = \langle 0,0,...,0\rangle$.

Theorem 2. *The vector operations have the following properties:*

(a) $\mathbf{u} \bullet \mathbf{u} \geq 0$*, and is 0 if and only if* $\mathbf{u} = \mathbf{0}$.

(b) $\mathbf{v} \bullet \mathbf{u} = \mathbf{u} \bullet \mathbf{v}$.

(c) $\mathbf{v} \bullet (\mathbf{u} + \mathbf{w}) = \mathbf{v} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{w}$.

(d) $r(\mathbf{u} \bullet \mathbf{v}) = (r\mathbf{u}) \bullet \mathbf{v}$.

(e) $r(\mathbf{u} + \mathbf{v}) = r\mathbf{v} + r\mathbf{u}$.

Problem 2. (a) Prove Theorem 2 for vectors in $\mathbb{R}^3$
(b) Prove Theorem 2 for vectors in $\mathbb{R}^n$

Theorem 3. *Let* $\mathbf{v}$ *be a vector in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$ *and define* $|\mathbf{v}|$ *to be* $\sqrt{\mathbf{v} \bullet \mathbf{v}}$. *Then* $|\mathbf{v}|$ *is the length of* $\mathbf{v}$. *If* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are nonzero vectors in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$ *and* θ *is the acute angle between them, then* $\mathbf{v}_1 \bullet \mathbf{v}_2 = |\mathbf{v}_1||\mathbf{v}_2| \cos\theta$. *In particular,*

$$\mathbf{v}_1 \perp \mathbf{v}_2 \Longleftrightarrow \mathbf{v}_1 \bullet \mathbf{v}_2 = 0.$$

Remark. Of course any vector $\mathbf{v}$ dotted with the zero vector $\mathbf{0}$ is zero, $\mathbf{v} \bullet \mathbf{0} = 0$. This may seem strange but we shall declare, by definition, that the zero vector is perpendicular to every vector. This common practice since it makes the statement of certain theorems cleaner.

Problem 3. (a) Prove Theorem 3 for vectors in $\mathbb{R}^2$
(b) Prove Theorem 3 for vectors in $\mathbb{R}^3$
Hint: Use the Law of Cosines.

Remark. For vectors in $\mathbb{R}^n$ the formulas in Theorem 3 are used to define length and angle in $\mathbb{R}^n$.

Problem 4. Let $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 1, 1, 0 \rangle$, and $\mathbf{w} = \langle 2, -2, 0 \rangle$.
(a) Compute each of the following expressions or explain why the expression is nonsense: (i) $\mathbf{v} \bullet (\mathbf{u} + 3\mathbf{w})$, (ii) $3\mathbf{u} + \mathbf{v} \bullet \mathbf{w}$, (iii) $(\mathbf{v} \bullet \mathbf{u} - \mathbf{w} \bullet \mathbf{w})^2$, (iv) $|\mathbf{v}| + |\mathbf{u}|$, (v) $|\mathbf{w}|\mathbf{v}$, (vi) $|\mathbf{u}||\mathbf{w}| + \mathbf{v}$.
(b) Find the angles between $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{w}$, and $\mathbf{v}$ and $\mathbf{w}$.

Problem 5. Let $\mathbf{u} = \langle 1, 2, 3, 4 \rangle$, and $\mathbf{v} = \langle 1, 1, 1, 1 \rangle$. Find the acute angle between them.

For vectors in $\mathbb{R}^2$ there is a useful graphical method of adding vectors. Suppose we want to add $\mathbf{v} = \langle 2, 1 \rangle$ and $\mathbf{u} = \langle 1, 2 \rangle$. Place an arrow with its tail at the origin and its head at $(2, 1)$ to represent $\mathbf{v}$. Next place an arrow with its tail at the origin and its head at $(1, 2)$ to represent $\mathbf{u}$. Now, move $\mathbf{u}$ so that its base is at the head of $\mathbf{v}$. The head of $\mathbf{u}$ is now located at the point $(3, 3)$. Draw an arrow with its tail at the origin and its head at $(3, 3)$. This new vector $\langle 3, 3 \rangle$ is just $\mathbf{v} + \mathbf{u}$. See Figure 1.

Subtraction is similar. To continue the same example, let $\mathbf{w} = \langle 3, 3 \rangle$. To find $\mathbf{w} - \mathbf{u}$, start with by drawing the arrow for $\mathbf{w}$. Then draw $-\mathbf{u}$ by flipping $\mathbf{u}$ through the origin, so that $-\mathbf{u}$ has its head at $(-1, -2)$. Move $-\mathbf{u}$ so that its tail is at the head of $\mathbf{w}$. Now the head of $\mathbf{u}$ is at $(2, 1)$ and we *see* that $\mathbf{w} - \mathbf{u} = \mathbf{v}$.

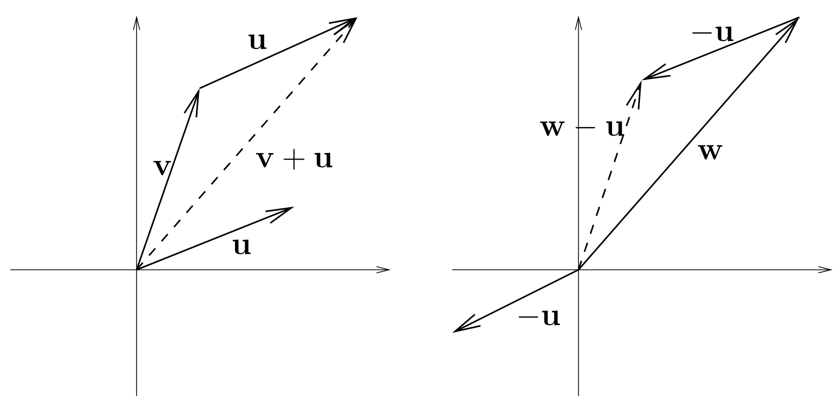

Figure 1: Vector addition

The scalar multiples of a vector are even easier to visualize. Figure 2 shows $3\mathbf{v}$, $-2\mathbf{v}$, and $\frac{1}{2}\mathbf{v}$.

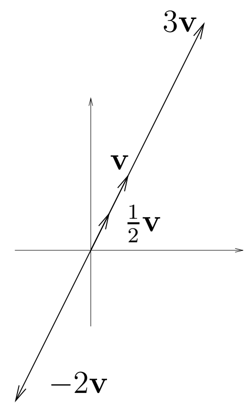

Figure 2: Scalar multiplies

0.3 Projections [Optional]

This material is used for the Gram-Schmidt Process and Least Squares Approximations.

Definition 4. Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in some $\mathbb{R}^m$. Let $\mathbf{w} = \dfrac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}}\mathbf{v}$. Then $\mathbf{w}$ is called the **projection** of $\mathbf{u}$ in the direction of $\mathbf{v}$ or **component** of $\mathbf{w}$ **paralell** to $\mathbf{v}$. Let $\mathbf{z} = \mathbf{u} - \mathbf{w}$. Then $\mathbf{z}$ is called the **component** of $\mathbf{u}$ **orthogonal** to $\mathbf{v}$.

Proposition 5. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be nonzero vectors in some* $\mathbb{R}^m$. *Let* $\mathbf{w} = \dfrac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}}\mathbf{v}$ *and* $\mathbf{z} = \mathbf{u} - \mathbf{w}$. *Then* $\mathbf{z} \bullet \mathbf{v} = 0$.

Proof. $\mathbf{z} \bullet \mathbf{v} = \mathbf{u} \bullet \mathbf{v} - \mathbf{w} \bullet \mathbf{v} = \mathbf{u} \bullet \mathbf{v} - \dfrac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}}\mathbf{v} \bullet \mathbf{v} = \mathbf{u} \bullet \mathbf{v} - \mathbf{u} \bullet \mathbf{v} = 0.$ □

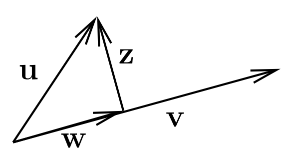

Figure 3: The components of **u** with respect to **v**

Example 2. The component of $\langle 1, 2, 4\rangle$ in the direction of $\langle 0, 0, 1\rangle$ is $\langle 0, 0, 4\rangle$ while the component of $\langle 1, 2, 4\rangle$ orthagonal to $\langle 0, 0, 1\rangle$ is $\langle 1, 2, 0\rangle$.

Problem 6. Find the components of $\langle 1, 2, 0, -1\rangle$ parallel and orthagonal to $\langle 1, 1, 1, 1\rangle$.

0.4 The Cross Product [Optional]

The cross product of two vectors in $\mathbb{R}^3$ is defined by

$$\langle a, b, c\rangle \times \langle x, y, z\rangle = \langle bz - cy, cx - az, ay - bx\rangle .$$

This may look arbitrary but is has several geometric properties that are especially useful in physics. It that setting this definition can be shown to be quite natural. However, there is no generalization of the cross product for vectors in $\mathbb{R}^n$, so we will not make much use of it.