

## NOTE ON A CLOSED SOLUTION CURVE

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ABSTRACT. Is topology a tool used in the study differential equations, or is the field of differential equations merely a branch of topology?

In Section 9.5 of [1], on predator-prey models, Example 1 analyses the system

$$(1) \quad \begin{aligned} dx/dt &= x - xy/2 \\ dy/dt &= -3y/4 + xy/4, \end{aligned}$$

where  $x \geq 0$  and  $y \geq 0$ . (We will always work in the first quadrant.) The critical points are  $(0, 0)$  (extinction) and  $(3, 2)$ . The linearization of the first is a saddle, of the second a center. While the origin must remain a saddle for the nonlinear system, the behavior around  $(3, 2)$  is not discernible just from the linearization. The text claims that it can be shown that the orbit is a closed curve for any initial condition off the axes and besides  $(3, 2)$ , but does not give a proof or justification. The aim of this note is to provide a justification that should be persuasive for undergraduates and point the reader toward the material needed for a rigorous proof.

For specificity, consider the initial condition  $(1, 1)$ . Our system (1) can be transformed to

$$dy/dx = \frac{-3y/4 + xy/4}{x - xy/2}.$$

This, as the text notes, is separable and has solution

$$3 \ln x + 4 \ln y - 2y - x = -3.$$

One can rewrite this as

$$x^3 y^4 = e^{x+2y-3}.$$

Let  $f(x, y) = x^3 y^4$  and  $g(x, y) = e^{x+2y-3}$ . Thus, our solution curve lines in the intersection of the surfaces  $z = f(x, y)$  and  $z = g(x, y)$ . Clearly,  $f = 0 < g$  on the coordinate axes.

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**Claim.** There exists an  $R > 0$  such that  $\sqrt{x^2 + y^2} \geq R$  implies  $f(x, y) < g(x, y)$ . *Proof.* In polar coordinates  $f = r^7 \cos^3 \theta \sin^4 \theta$  and  $g = e^{r(\cos \theta + 2 \sin \theta) - 3}$ . Let  $\alpha = \max\{\cos^3 \theta \sin^4 \theta : 0 \leq \theta \leq \pi/2\} > 0$ ; and let  $\beta = \min\{\cos \theta + 2 \sin \theta : 0 \leq \theta \leq \pi/2\} > 0$ . By L'Hospital's Rule

$$\lim_{r \rightarrow \infty} \frac{r^7 \alpha}{e^{r\beta - 3}} = 0.$$

Hence, there exists an  $R > 0$  such that  $r \geq R$  implies  $r^7 \alpha / e^{r\beta - 3} < 1$ . Thus, for  $\sqrt{x^2 + y^2} = r \geq R$  we have

$$f(x, y) \leq r^7 \alpha < e^{r\beta - 3} \leq g(x, y),$$

as claimed.

This shows that these surfaces meet only for values of  $x$  and  $y$  inside a closed bounded region (a quarter disk). Since,  $f(3, 2) > g(3, 2)$ , the intersection is not empty. One can use transverse intersection theory (see references below) to argue that the intersection of these surfaces is a finite union of simple closed curves, or loops. Any such loop must project to a loop in the domain, since the surfaces are graphs of functions. A loop in the domain bounds a disk (by the Jordan Curve Theorem) over which the surface  $z = f - g$  must have an extrema. But  $f - g$  only has one extrema,  $(3, 2)$ , a local minimum, as is easily checked. Thus, our solution curve is a loop going around this point.

Transverse intersection theory is a topic in differential topology. The key result needed to make the discussion above formal is Sard's Theorem. Two standard references, which could be read with some profit by advanced undergraduates, are [2] and [3].

Thus, this is a good place to point out to students in a differential equations course that it was questions of this type that led to the birth of topology. Since the equation turned out to be separable this point could be made very early in a differential equations course, or even in a calculus course. One could perhaps hope that those who teach introductory topology courses would make such connections.

#### REFERENCES

- [1] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 7th edition, Wiley High Education, 2001.
- [2] Gullemin and Pollack, *Differential Topology*, Prentice-Hall, Inc, Englewood Cliffs, New Jersey, 1974.
- [3] Hirsch, *Differential Topology*, Springer-Verlag, New York, New York, 1976.

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