Integrating Powers of the Secant Function

Your book is a bit vague on how to do $\int sec^n(x)\,dx$. We dived the problem into two cases: n is even, or n is odd. The first case is easy. We use the "shortcut notations" $t = \tan(x)$, and $s = \sec(x)$, throughout. Then $dt = s^2\,dx$, and $ds = st\,dx$. Recall $t^2 + 1 = s^2$.

Case I: n is even. Let $n = 2k$. Then

$$\int s^{2k}\,dx = \int s^{2k-2}\,s^2 dx = \int (s^2)^{k-1}\,s^2 dx = \int (t^2+1)^{k-1}\,dt$$

The last integral is just the integral of a polynomial.

Example:

$$\int s^6\,dx = \int s^4\,s^2 dx = \int (t^2+1)^2\,dt = \int t^4 + 2t^2 + 1\,dt = \frac{t^5}{5} + \frac{2t^3}{3} + t + C$$

Case II: n is odd. This case is much harder. But, we cam make it simpler by first deriving a **reduction formula**. Recall that $\int s\,dx = \ln|s+t| + C$. Let $n = 2k+1$, with $k \geq 1$. Let

$$I = \int s^{2k+1}\,dx = \int s^{2k-1}\,s^2 dx.$$

This time we use integration by parts. Let $u = s^{2k-1}$ and $dv = s^2 dx$. Then $du = (2k-1)s^{2k-2}\,st dx = (2k-1)s^{2k-1}t\,dx$ (recall $s' = st$), and $v = t$.

Now,

$$\begin{aligned}
I &= ts^{2k-1} - (2k-1)\int s^{2k-1}t^2\,dx \\
&= ts^{2k-1} - (2k-1)\int s^{2k-1}(s^2-1)\,dx \\
&= ts^{2k-1} - (2k-1)\int s^{2k+1} - s^{2k-1}\,dx \\
&= ts^{2k-1} - (2k-1)\left(\int s^{2k+1}\,dx - \int s^{2k-1}\,dx\right) \\
&= ts^{2k-1} - (2k-1)\int s^{2k+1}\,dx + (2k-1)\int s^{2k-1}\,dx \\
&= ts^{2k-1} - (2k-1)I + (2k-1)\int s^{2k-1}\,dx
\end{aligned}$$

Thus,

$$\begin{aligned}
I + (2k-1)I &= ts^{2k-1} + (2k-1)\int s^{2k-1}\,dx \\
2kI &= ts^{2k-1} + (2k-1)\int s^{2k-1}\,dx
\end{aligned}$$

So,

$$\int s^{2k+1}\,dx = \frac{ts^{2k-1} + (2k-1)\int s^{2k-1}\,dx}{2k}.$$

This is called a “reduction formula” since the power of the secant function has been reduced. Now let’s see how to use this.

Example: We compute $\int \sec^7(x)\,dx$. First $7 = 2 \cdot 3 + 1$, so we use $k = 3$.

$$\begin{aligned}\int s^7\,dx &= \frac{ts^5 + 5\int s^5\,dx}{6}\\ \int s^5\,dx &= \frac{ts^3 + 3\int s^3\,dx}{4}\\ \int s^3\,dx &= \frac{ts + \int s\,dx}{2}\\ \int s\,dx &= \ln|s+t| + C\end{aligned}$$

Now put these together to get,

$$\int s^7\,dx = \frac{1}{6}ts^5 + \frac{5}{6\cdot 4}ts^3 + \frac{5\cdot 3}{6\cdot 4\cdot 2}ts + \frac{5\cdot 3}{6\cdot 4\cdot 2}\ln|s+t| + C$$

Study this closely. Do you see the pattern?

Example: We do $\int \sec^{11}(x)\,dx$ just from the pattern.

$$\int s^{11}\,dx = \frac{1}{10}ts^9 + \frac{9}{10\cdot 8}ts^7 + \frac{9\cdot 7}{10\cdot 8\cdot 6}ts^5 + \frac{9\cdot 7\cdot 5}{10\cdot 8\cdot 6\cdot 4}ts^3 + \frac{9\cdot 7\cdot 5\cdot 3}{10\cdot 8\cdot 6\cdot 4\cdot 2}ts$$
$$+\frac{9\cdot 7\cdot 5\cdot 3}{10\cdot 8\cdot 6\cdot 4\cdot 2}\ln|s+t| + C$$

Integrating Products of Powers of Tangent and Secant

We now move on to integrals of the form $\displaystyle\int \tan^m(x)\sec^n(x)\,dx$.

Case I: n is even. Let $n = 2k$. The same method we used for $\displaystyle\int s^{2k}\,dx$ works.

Example:

$$\int t^{13}s^6\,dx = \int t^{13}s^4\,s^2dx = \int t^{13}(t^2+1)^2\,dt = \int t^{13}(t^4+2t^2+1)\,dt =$$
$$\int t^{17} + 2t^{15} + t^{13}\,dt = \frac{t^{18}}{18} + \frac{2t^{16}}{16} + \frac{t^{14}}{14} + C$$

Case II: n is odd and m is odd. This is easy. Let $m = 2p+1$ and $n = 2k+1$. Recall $tsdx = ds$.

$$\int t^{2p+1}s^{2k+1}\,dx = \int t^{2p}s^{2k}\,tsdx = \int (s^2-1)^p s^{2k}ds,$$

which is a polynomial in secant, integrated against secant.

Example:

$$\int t^7s^5\,dx = \int t^6s^4\,stdx = \int (s^2-1)^3s^4\,ds = \int (s^6 - 3s^4 + 3s^2 - 1)s^4\,ds$$
$$= \int s^{10} - 3s^8 + 3s^6 - s^4\,ds = \frac{s^{11}}{11} - \frac{s^9}{3} + \frac{3s^7}{7} - \frac{s^5}{5} + C$$

Case III: n is odd, and m is even. Let $m = 2p$. Now,

$$\int t^{2p} s^n \, dx = \int (s^2 - 1)^p s^n \, dx,$$

and so we have only powers of secant; we know how to do these. But, even simple examples can be quite messy.

Example: Suppose $m = 2$ and $n = 3$. Then

$$\int t^2 s^3 \, dx = \int (s^2 - 1) s^3 \, dx = \int s^5 - s^3 \, dx.$$

Using the reduction formula we get

$$\begin{aligned} \int s^5 \, dx &= \frac{1}{4} t s^3 + \frac{3}{4 \cdot 2} t s + \frac{3}{4 \cdot 2} \ln |s + t| + C \\ \int s^3 \, dx &= \frac{1}{2} t s + \frac{1}{2} \ln |s + t| + C \end{aligned}$$

Subtracting gives

$$\frac{1}{4} t s^3 - \frac{1}{8} t s - \frac{1}{8} \ln |s + t| + C$$

as the final answer. (Hopefully you see why the C's do not cancel.)

Extra Credit Project

THIS WILL COUNT AS HALF A TEST GRADE. YOU MAY NOT WORK WITH ANYONE ELSE.

Project: Write a computer program whose input is a pair of nonnegative integers, m and n. The output is to be the integral of $t^m s^n$. You can use any programming language, but do not use any higher level packages or library functions. Turn in your program on a disk along with a printout of several examples. (Pretend I am a potential customer and you and trying to sell me your product.)

Due Date: October 20.

Hint: For expanding terms like $(a + b)^k$ you need the Binomial Theorem,

$$(a + b)^k = \sum_{n=0}^{k} \frac{k!}{n!(k-n)!} a^{k-n} b^n.$$

For example, $(a + b)^4 = a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$.