Derivation of the Trapezoidal Rule Error Estimate

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Theorem: Let $f(x)$ be a twice differentiable function on the interval $[a, b]$. Let

$$I = \int_a^b f(x)\, dx$$

and

$$T(n) = \frac{b-a}{2n}\left(f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$$

where $x_i = a + i\frac{b-a}{n}$. Let

$$M = \max_{a \le x \le b}\{|f''(x)|\}.$$

Then

$$|I - T(n)| \le \frac{(b-a)^3}{12n^2}M.$$

Proof: We shall break the proof up into several steps. The first one is the most essential and the most difficult. Its proof will use both the Fundamental Theorem of Calculus and the Mean Value Theorem. It is nice to see such theoretical tools used to solve a practical problem.

STEP 1: Assume for now that $a = 0$, $b = 1$ and $f(a) = f(b) = 0$. Let M be as in the theorem. We will prove that

$$\left|\int_0^1 f(x)\, dx\right| \le M/12.$$

Proof of Step 1: Let $g(x) = \frac{M}{2}x(1-x)$. Then $g(0) = g(1) = 0$. Also, the reader can check that

$$\left|\int_0^1 g(x)\, dx\right| = M/12,$$

and

$$|g''(x)| = M.$$

We will show that $|f(x)| \le g(x)$ on $[0, 1]$. The desired conclusion follows.

Suppose however that this is false, that there is a number $q \in [0, 1]$ for which $f(q) > g(q)$. (The other case, $f(q) < -g(q)$, is similar.) Our strategy will be to show that there are real numbers s and t with $s < t$ such that $f'(s) > g'(s)$ and $f'(t) < g'(t)$. See figure. We will then apply the Mean Value Theorem to f' and g' and use facts about g to force $|f''(x)| > M$, for some value of x. This, of course, contradicts a hypothesis of the Theorem. Thus, if the hypotheses are true then, $|f(x)| \le g(x)$ on $[0, 1]$, is also true.

We proceed. By the Fundamental Theorem of Calculus

$$\int_0^q f'(x)\, dx = f(q) - f(0) = f(q),$$

and

$$\int_0^q g'(x)\, dx = g(q) - g(0) = g(q).$$

Since $f(q) > g(q)$ we have

$$\int_0^q f'(x)\,dx > \int_0^q g'(x)\,dx.$$

Hence there is an $s \in [0,q]$ for which $f'(s) > g'(s)$. Again by the Fundamental Theorem of Calculus

$$\int_q^1 f'(x)\,dx = f(1) - f(q) = -f(q),$$

and

$$\int_q^1 g'(x)\,dx = g(1) - g(q) = -g(q).$$

Since $-f(q) < -g(q)$ we have

$$\int_q^1 f'(x)\,dx < \int_q^1 g'(x)\,dx.$$

Hence there is a $t \in [q,1]$ for which $f'(t) < g'(t)$.

Clearly, $s \neq t$ so, $s < t$. The reader can check that $g'(s) > g'(t)$. (Note that $g(x)$ is a downward opening parabola so the derivative is decreasing.)

We can now report that

$$f'(s) > g'(s) > g'(t) > f'(t).$$

It follows that

$$\left|\frac{f'(t) - f'(s)}{t - s}\right| > \left|\frac{g'(t) - g'(s)}{t - s}\right|.$$

By the Mean Value Theorem there are numbers r_1 and r_2 in between s and t where

$$f''(r_1) = \frac{f'(t) - f'(s)}{t - s}$$

and

$$g''(r_2) = \frac{g'(t) - g'(s)}{t - s}.$$

Then $|f''(r_1)| > |g''(r_2)|$. But, $|g''(r_2)| = M$ since $g''(x)$ is always equal to $-M$. Thus $|f''(r_1)| > M$, which as we noted above is a contradiction and completes the proof of Step 1.

Discussion: It is easy to lose sight of what was accomplished in Step 1 amid the vast array of details. The point is this. For the Trapezoidal Rule with $n = 1$, $T(1) = 0$ for the function in Step 1. Thus the error is $|I|$ itself. We showed that the bound on the magnitude of the second derivative of $f(x)$ forces $f(x)$ to bounded by the parabolas $\pm g(x)$ on $[0,1]$. This gave a bound on $|I|$.

In Steps 2 & 3 we generalize the result in Step 1 by first allowing for arbitrary intervals and then by letting the function take nonzero values at the end points. Finally in Step 4 we apply Step 3 to each of the n subintervals, the partition elements, used in the Trapezoidal Rule and then take the sum.

STEP 2: Now we let a and b be arbitrary (with $a < b$). We still assume $f(a) = f(b) = 0$. M is as before. Then we will prove that

$$\left|\int_a^b f(x)\,dx\right| \le (b-a)^3 \frac{M}{12}.$$

Proof of Step 2: Let $g(x) = f(a + (b-a)x)$. Notice that $g(0) = f(a) = 0$ and $g(1) = f(b) = 0$. Also $g''(x) = (b-a)^2 f''(a + (b-a)x)$. Now for $0 \le x \le 1$ we have $a \le a - (b-a)x \le b$. Thus, $|g''(x)| \le (b-a)^2 M$, for $x \in [0.1]$.

Now, by Step 1 we know that

$$\left| \int_0^1 g(x)\, dx \right| \le (b-a)^2 \frac{M}{2}.$$

But

$$\int_0^1 g(x)\, dx = \int_0^1 f(a + (b-a)x)\, dx.$$

Let $u(x) = a + (b-a)x$. Then $du = (b-a)dx$ and $u(0) = a$ and $u(1) = b$. So,

$$\int_0^1 f(a + (b-a)x)\, dx = \frac{1}{b-a} \int_{u(0)}^{u(1)} f(u)\, du = \frac{1}{b-a} \int_a^b f(u)\, du.$$

Hence,

$$\left| \int_a^b f(u)\, du \right| = (b-a) \left| \int_0^1 g(x)\, dx \right| \le \frac{(b-a)^3}{12} M.$$

This completes the proof of Step 2.

STEP 3: Now suppose $f(a) = A$ and $f(b) = B$. Let $g(x)$ be the line segment connecting (a, A) and (b, B). Then

$$\left| \int_a^b f(x) - g(x)\, dx \right| \le (b-a)^3 \frac{M}{12}.$$

Proof of Step 3: Let $h(x) = f(x) - g(x)$. Then $h(a) = A - A = 0$ and $h(b) = B - B = 0$. Also, $h''(x) = f''(x) + g''(x) = f''(x) + 0 = f''(x)$. Apply Step 2 to $h(x)$. This completes the proof of Step 3.

STEP 4: Now we put it all together and prove the theorem. Let

$$g_i(x) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i) + f(x_i).$$

Don't be too put off. This cumbersome expression is just the equation of the line segment joining $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$. The area between $g_i(x)$ and the x-axis from x_i and x_{i+1} is the trapezoid used in the Trapezoidal Rule. Thus,

$$T(n) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} g_i(x)\, dx.$$

Thus,

$$
\begin{aligned}
|I - T(n)| &= \left| \int_a^b f(x)\,dx - \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} g_i(x)\,dx \right| \\
&= \left| \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,dx - \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} g_i(x)\,dx \right| \\
&= \left| \sum_{i=0}^{n-1} \left[\int_{x_i}^{x_{i+1}} f(x)\,dx - \int_{x_i}^{x_{i+1}} g_i(x)\,dx \right] \right| \\
&= \left| \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x) - g_i(x)\,dx \right| \\
&\leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x) - g_i(x)\,dx \right| \\
&\leq \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^3}{12} M \\
&= \sum_{i=0}^{n-1} \left(\frac{b-a}{n} \right)^3 \frac{M}{12} \\
&= n \left(\frac{b-a}{n} \right)^3 \frac{M}{12} \\
&= \frac{(b-a)^3}{12n^2} M.
\end{aligned}
$$

This completes the proof of the Theorem.

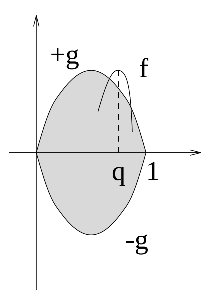

Figure 1