Print Name: Key

Math 250 **Test 2** **Scientific Calculators Allowed**

1. [5 points] Compute the sum $\sum_{i=3}^{7} i^2$. $= 9+16+25+36+49 = 135$

2. [10 points] Compute $\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx$. Simplify your answer.

$u = \sin x \quad du = \cos x \, dx \quad \cos^2 x = 1 - \sin^2 x.$

Thus, $\int_0^1 u^2(1-u^2)\,du = \int_0^1 u^2 - u^4 \, du = \frac{u^3}{3} - \frac{u^5}{5}\Big|_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$

3. [20 points] Compute the series or show it diverges.

(a) $\sum_{n=1}^{\infty} \frac{2}{(-3)^{n-1}}$ $\quad a = 2$, $r = -\frac{1}{3}$

$\frac{2}{1-(-\frac{1}{3})} = \frac{2}{\frac{4}{3}} = \frac{3}{2}$

(b) $\sum_{n=0}^{\infty} \frac{1}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n-1}$ $\quad a = 1$, $r = \frac{1}{4}$

$\frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$

(c) $\sum_{n=1}^{\infty} \frac{5^n}{4^{n+1}} = \sum_{n=1}^{\infty} \frac{5}{16}\left(\frac{5}{4}\right)^{n-1}$

$r = \frac{5}{4} > 1$,

diverge

(d) $\sum_{j=1}^{\infty} \frac{1}{(j+1)(j+2)} = \sum_{j=1}^{\infty} \frac{1}{j+1} - \frac{1}{j+2}$

$= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \cancel{\cdots}$

$= \frac{1}{2}$

4. [10 points] For each series prove whether it converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$a_n = \frac{n}{n+1} \to 1$ not 0.

This series diverges by the Test for Divergance

(b) $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$

$\frac{1}{n^3+1} < \frac{1}{n^3}$

So, this series converges by comparison with a convergent p-series.

5. [10 points] Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ convergres or diverges. Hint: use integral test.

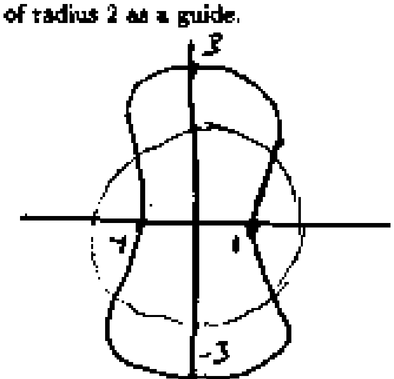
$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \int \frac{1}{u}\, du = \ln|u| = \ln|\ln x| \Big|_2^{\infty} = \text{diverges since } \lim_{x\to\infty} \ln x = \infty.$$

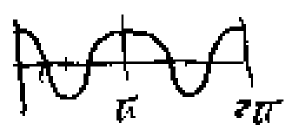
Let $u = \ln x$.

$du = \frac{1}{x} dx$

The Series ~~converg~~ diverges by the Integral Test.

6. [20 points] (a) Graph $r = 2 - \cos 2\theta$ is polar coordinates. Label the x and y intercepts. Hint: Draw a faint circle of radius 2 as a guide.





(b) Find the area enclosed.

$$\frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \frac{1}{2}\int_0^{2\pi} (2-\cos 2\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} 4 - 4\cos 2\theta + \cos^2 2\theta\, d\theta$$

$$= \frac{1}{2}\left(8\pi - 0 + \frac{2\pi}{2}\right) = \frac{9\pi}{2}$$

7. [10+5 points] (a) Find the arc length of the curve determined by $x(t) = 4t^{3/2}$ and $y(t) = 2t + 1$ from $t = 1$ to $t = 3$.

$x' = \frac{3}{2} t^{\frac{1}{2}}$ $\quad y' = 2$

$(x')^2 = \frac{9}{4} t$ $\quad (y')^2 = 4$

$L = \int_1^3 \sqrt{36t + 4}\, dt$

$u = 36t + 4 \quad du = 36\, dt$

$\frac{1}{36} \int_{40}^{112} \sqrt{u}\, du = \frac{1}{36} \frac{2}{3} u^{3/2} \Big|_{40}^{112}$

$\frac{1}{54} \left((112)^{3/2} - (40)^{3/2} \right) \approx 17.27$

(b) Find the equation in slope-intersecpt form for the line tangant to the curve determined by $x(t) = \sec^2 t$ and $y(t) = \sin 3t$ at $t = \pi/3$.

$x(\pi/3) = \frac{1}{\cos^2(\frac{\pi}{3})} = \frac{1}{(\frac{1}{2})^2} = 4$

$y(\pi/3) = \sin(\pi) = 0$

$x' = 2 \sec t \cdot \sec t \tan t$

$x'(\frac{\pi}{3}) = 2 \cdot 2 \cdot 2 \cdot \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = 8\sqrt{3}$

$y' = 3 \cos(3t)$

$y'(\frac{\pi}{3}) = 3 \cos(\pi) = -3$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-3}{8\sqrt{3}} = \frac{-\sqrt{3}}{8}$

$y - 0 = \frac{-\sqrt{3}}{8} (x - 4)$

$\boxed{y = -\frac{\sqrt{3}}{8} x + \frac{\sqrt{3}}{2}}$

8. [10 points] Prove that $s_k = \sum_{n=1}^{k} ar^{n-1} = \frac{a(1 - r^k)}{1 - r}$ for all $r \neq 1$.

$S_k = a + ar + ar^2 + ar^3 + \cdots + ar^{k-1}$

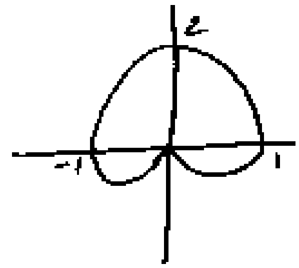
$rS_k = \quad ar + ar^2 + ar^3 + \cdots + ar^{k-1} + ar^k$

$S_k - rS_k = a - ar^k$

$(1 - r) S_k = a(1 - r^k)$

$S_k = \frac{a(1 - r^k)}{1 - r}$ for $r \neq 1$.

9. [20 BONUS points] Consider the cardioid $r = 1 + \sin\theta$. Rotate it about the y-axis.
(a) Set up an integral to find the surface area.
(b) Evaluate the integral.



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi x \sqrt{r^2 + (r')^2}\ d\theta$$

$$= 2\pi \int \underset{1+\sin\theta}{r} \cos\theta \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\ d\theta$$

$$= 2\pi \int (1+\sin\theta)\cos\theta \sqrt{1 + 2\sin\theta + \underbrace{\sin^2\theta + \cos^2\theta}_{1}}\ d\theta$$

$$= 2\pi \int (1+\sin\theta) \cos\theta \sqrt{2 + 2\sin\theta}\ d\theta$$

$$= 2\pi\sqrt{2} \int (1+\sin\theta)^{3/2} \cos\theta\, d\theta \qquad u = 1+\sin\theta \quad du = \cos\theta\, d\theta$$

$$= 2\pi\sqrt{2} \int_0^2 u^{3/2}\, du = 2\pi\sqrt{2}\ \frac{2}{5} u^{\frac{5}{2}} \Big|_1^2 =$$

$$2\pi\sqrt{2} \cdot \frac{2}{5} \cdot (2)^{5/2} = \frac{32\pi}{5}$$