

Some Notation From Set Theory for Calculus Students

A **set** is a collection of **elements**. The expression “ $p \in S$ ” means p is an element of the set S . A set may be defined in several ways: in ordinary English, *e.g.*, let A be the set of positive even integers; by listing its elements within braces, *e.g.*, let $A = \{2, 4, 6, 8, \dots\}$; or by using “set builder” notation, *e.g.*, $A = \{n \in \mathbb{Z} \mid n > 0 \text{ and } n \text{ is even}\}$, read, A is the set of all integers n such that $n > 0$ and n is even (\mathbb{Z} is the standard notation for the integers).

A set does not have an order. Thus $\{a, b\} = \{b, a\}$. An **ordered set** is a set together with an ordering. When we want to stress that a set has been endowed with an ordering we will use parentheses instead of braces: (a, b) is an ordered set and is not equal to (b, a) .

The following notations are standard.

- $\phi = \{\}$, the empty set.
- $A \subset B$: read A is a subset of B , means every element of A is an element of B . *Example:* $\{2, 5\} \subset \{1, 2, 3, 4, 5\}$.
- $A \cup B$: read A union B , means the set of all elements that are in A **or** in B . *Example:* $\{\$, *, !\} \cup \{\alpha, !, *, 17\} = \{\$, *, !, \alpha, *, 17\}$.
- $A \cap B$: read A intersection B , means the set of all elements that are in A **and** in B . *Example:* $\{\$, *, !\} \cap \{\alpha, !, *, 17\} = \{!\}$.
- $A - B$: read A minus B , means the set of all elements of A that are not elements of B . *Example:* $\{\$, *, !\} - \{\alpha, !, *, 17\} = \{\$, *\}$.
- $A \times B$: read A cross B , means the set of ordered pairs (a, b) where $a \in A$ and $b \in B$. Since there is a natural one-to-one correspondence between $(A \times B) \times C$ and $A \times (B \times C)$, $((a, b), c) \longleftrightarrow (a, (b, c))$, we shall ignore the distinction between them and use the notation $A \times B \times C$ for the set $\{(a, b, c) \mid a \in A, b \in B, \text{ and } c \in C\}$. Other multiple cross products are defined similarly. *Examples:* $\{1, 3\} \times \{0, 1, 2\} = \{(1, 0), (1, 1), (1, 2), (3, 0), (3, 1), (3, 2)\}$. $\{*, \#\} \times \{\%\} = \{(*, \%), (\#, \%)\}$.
- $A^n = A \times \dots \times A$, n times. *Example:* $\{2, 3\}^3 = \{(2, 2, 2), (2, 2, 3), (2, 3, 2), (2, 3, 3), (3, 2, 2), (3, 2, 3), (3, 3, 2), (3, 3, 3)\}$.

Some standard sets are:

- \mathbb{Z} : the integers (most likely from the German *Zahl*, meaning number),
- \mathbb{Q} : the rational numbers (quotients),
- \mathbb{R} : the real numbers, and
- \mathbb{C} : the complex numbers.

Remark: The sets \mathbb{Z} , \mathbb{Q} , and \mathbb{R} are normally given an ordering. Interestingly, \mathbb{C} is not typically ordered. Interval Notation.

$$\begin{array}{ll}
 [a, b] & = \{x \in \mathbb{R} \mid a \leq x \leq b\} & [a, \infty) & = \{x \in \mathbb{R} \mid a \leq x\} \\
 (a, b) & = \{x \in \mathbb{R} \mid a < x < b\} & (a, \infty) & = \{x \in \mathbb{R} \mid a < x\} \\
 (a, b] & = \{x \in \mathbb{R} \mid a < x \leq b\} & (-\infty, b] & = \{x \in \mathbb{R} \mid x \leq b\} \\
 [a, b) & = \{x \in \mathbb{R} \mid a \leq x < b\} & (-\infty, b) & = \{x \in \mathbb{R} \mid x < b\}
 \end{array}$$

Remark: The notation “ (a, b) ” is ambiguous; it could represent an interval or an ordered pair. One has to consider the context to understand the intended meaning. On behalf of mathematicians everywhere I apologize for any inconvenience this may cause.

Examples:

- $(-\infty, \sqrt{7}] \cup [\sqrt{7}, \infty) = \{x \in \mathbb{R} \mid x \leq -\sqrt{7}\} \cup \{x \in \mathbb{R} \mid x \geq \sqrt{7}\}$ is the solution set for $x^2 - 7 \geq 0$.

- $(-\infty, 0) \cup (0, \infty) = \mathbb{R} - \{0\}$ is the natural domain of $1/x$.
- \mathbb{R}^2 is the plane. \mathbb{R}^3 is 3-dimensional space. \mathbb{R}^4 is 4-dimensional space. And so on.
- $\phi \subset A$, $\phi = A \cap \phi$, and $A = A \cup \phi$ are true statements for all sets A .
- $\{x \in \mathbb{R} \mid -2 \leq x < 5\} = [-2, 5) = [-2, 7] \cap (-10, 5)$.
- $S = [0, 1] \times [0, 1]$ is the *unit square* in the plane \mathbb{R}^2 with corners $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$.

Quantifiers: The symbols \forall and \exists are rather handy. \forall means “for all.” \exists means “there exists.” They are called *quantifiers* and are commonly used in logic.

Examples:

- $\forall x \geq 0 \exists y \geq 0$ such that $y^2 = x$. This means, every nonnegative real number has a nonnegative square root.
- A function f has a *relative maximum* at c if $\exists \epsilon > 0$ such that $\forall x \in (c - \epsilon, c + \epsilon) - \{c\}$ we have $f(x) < f(c)$.
- A function f is *unbounded from above* if $\forall B > 0 \exists x \in \mathbb{R}$ such that $f(x) > B$.

Problems: (These are not due. See me if you have questions.)

1. Describe $[0, 1] \times [0, 2] \times [0, 3]$.
2. Simplify $((1, 3) \cap (2, 5)) \cup [3, 4)$.
3. Let $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9\}$, $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$, and $C = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. Draw $A - B$, $A - C$, $A \cap C$, $(A - B) \cap C$, and $A - (B \cap C)$.
4. Find the solution set in \mathbb{R}^2 of $\sin x \cos y = 0$.
5. Draw $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times \mathbb{R}$, and $((0, 1] \cup \{2, 3\}) \times ([-2, -1] \cup (2, 3))$ as subsets of \mathbb{R}^2 .
6. Let A be a set. What is $A \times \phi$?
7. Let A , B , and C be sets. Prove that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. (You can draw pictures to “see” this, but you need to reason from the definitions to prove it.)
8. (a) Write a definition for a point to be a relative minimum of a function using quantifiers.
 (b) Write a definition for a function to be unbounded from below using quantifiers.
 (c) Translate “ $\forall \delta > 0 \exists N \in \mathbb{Z}$ such that \forall integers $n > N$ we have $0 < \frac{1}{n} < \delta$ ” into English. Is it a true statement?