Vectors Calculus Summary [Draft][1]

Types of Integrals. We can integrate along a curve C, over a surface S and through a 3-dimension region V. If the integrand is 1 we get arc length, surface area and volume respectively. If we are given some "density" function we can find the "mass".

Example: Let C be given by $r(t) = \langle t, t^2, t^3 \rangle$ for $0 \le t \le 1$. Let S be the portion of $z = x^2 + 2y + 3$ over the unit disk U. Let V be the region inside the cylinder $x^2 + y^2 = 1$, below S and above the xy-plane. Find the length of C, the area of S and the volume of V.

$$\text{Length} = \int_C ds = \int_0^1 |r'(t)|\, dt = \int_0^1 \sqrt{1 + 4t^2 + 9t^4}\, dt \approx 1.863022983.$$

$$\text{Surface area} = \iint_S dS = \iint_U \sqrt{1 + (z_x)^2 + (z_y)^2}\, dA =$$

$$\int_0^{2\pi} \int_0^1 \sqrt{1 + (2x)^2 + 2^2}\; r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 \sqrt{5 + 4r^2 \cos^2\theta}\, r\, dr\, d\theta \approx 7.670233535.$$

$$\text{Volume} = \iiint_V dV = \int_0^{2\pi} \int_0^1 \int_0^{r^2\cos^2\theta + 2r\sin\theta + 3} r\, dz\, dr\, d\theta =$$

$$\int_0^{2\pi} \int_0^1 (r^2\cos^2\theta + 2r\sin\theta + 3)\, r\, dr\, d\theta = \frac{13}{4}\pi.$$

Example: Use the same C, S and V. Let $h(x, y, z) = xyz$ be a density function. Find the masses of C, S and V.

$$\text{Mass of C} = \int_C h\, ds = \int_0^1 h(t, t^2, t^3)|r'(t)|\, dt = \int_0^1 t^6 \sqrt{1 + 4t^2 + 9t^4}\, dt \approx 0.442101217$$

$$\text{Mass of S} = \iint_S h\, dS = \iint_U h(x, y, x^2 + 2y + 3)\sqrt{1 + (z_x)^2 + (z_y)^2}\, dA =$$

$$\int_0^{2\pi} \int_0^1 h(r\cos\theta, r\sin\theta, r^2\cos^2\theta + 2r\sin\theta + 3)\sqrt{5 + 4r^2\cos^2\theta}\; r\, dr\, d\theta =$$

$$\int_0^{2\pi} \int_0^1 r\cos\theta r\sin\theta (r^2\cos^2\theta + 2r\sin\theta + 3)\sqrt{5 + 4r^2\cos^2\theta}\; r\, dr\, d\theta =$$

$$\int_0^1\int_0^{2\pi} r^4\cos^3\theta\sin\theta\sqrt{*}+2r^3\cos\theta\sin^2\theta\sqrt{*}+3r^2\cos\theta\sin\theta\sqrt{*}\;d\theta dr =$$

$$\int_0^1\int_0^{2\pi} r^2\cos\theta\sin\theta(r^2\cos^2\theta+3)\sqrt{*}+2r^3\cos\theta\sin^2\theta\sqrt{*}\;dr\,d\theta = 0.$$

To see this let (or if you don't want to see this, skip to the "Mass of V")

$$f(\theta)=\cos\theta\sin\theta E(\cos\theta) \quad\text{and}\quad g(\theta)=\cos\theta\sin^2\theta E(\cos\theta),$$

where E is any even integrable function. Check that $g(\theta+\pi)=-g(\theta)$. Then

$$\int_0^{2\pi} g(\theta)\,d\theta=\int_0^{\pi} g(\theta)\,d\theta+\int_\pi^{2\pi} g(\theta)\,d\theta=\int_0^{\pi} g(\theta)\,d\theta+\int_0^{\pi} g(\theta+\pi)\,d\theta=\int_0^{\pi} g(\theta)\,d\theta-\int_0^{\pi} g(\theta)\,d\theta=0.$$

Next check that $f(\theta+\pi)=f(\theta)$ and $f(\pi-\theta)=-f(\theta)$. Now

$$\int_0^{2\pi} f(\theta)\,d\theta=2\int_0^{\pi} f(\theta)\,d\theta=2\left(\int_0^{\pi/2} f(\theta)\,d\theta+\int_{\pi/2}^{\pi} f(\theta)\,d\theta\right).$$

But, letting $\phi=\pi-\theta$ we have

$$\int_{\pi/2}^{\pi} f(\theta)\,d\theta=-\int_{\pi/2}^{0} f(\pi-\phi)\,d\phi=\int_0^{\pi/2} f(\pi-\phi)\,d\phi=-\int_0^{\pi/2} f(\phi)\,d\phi,$$

and the result follows.

$$\text{Mass of V}=\iiint_V h(x,y,z)\,dV=\int_0^{2\pi}\int_0^1\int_0^{r^2\cos^2\theta+2r\sin\theta+3} h(r\cos\theta,r\sin\theta,z)\,r\,dz\,dr\,d\theta=$$

$$\int_0^{2\pi}\int_0^1\int_0^{r^2\cos^2\theta+2r\sin\theta+3}(r^2\cos\theta\sin\theta)z\,r\,dz\,dr\,d\theta=$$

$$\int_0^{2\pi}\int_0^1(r^2\cos\theta\sin\theta)(r^2\cos^2\theta+2r\sin\theta+3)\,r\,dr\,d\theta=$$

$$\int_0^{2\pi}\int_0^1 r^5\cos^3\theta\sin\theta+2r^4\cos\theta\sin^2\theta+3r^3dr\,d\theta=$$

$$\int_0^{2\pi}\frac{1}{6}\cos^3\theta\sin\theta+\frac{2}{5}\cos\theta\sin^2\theta+\frac{3}{4}\,d\theta=0+0+\frac{3}{4}2\pi=\frac{3\pi}{2}$$

Integrals of Vector Fields: Work & Flux. If F is a vector field we can compute the work done as we travel along a curve C by taking the integral of the component of F tangent to C along C.

$$\text{Work} = \int_C F \bullet T\, ds.$$

We can compute the flux of F through S by using the surface integral over S of the component of F normal to S.

$$\text{Flux} = \iint_S F \bullet N\, dS.$$

Example: Using the same S and C as before and $F = \langle 3x + y, z, z^2 \rangle$ find the work done along C and the flux through S.

$$\text{Work} = \int_C F \bullet T\, ds = \int_0^1 \left\langle 3t + t^2, t^3, t^6 \right\rangle \bullet \frac{r'(t)}{|r'(t)|} |r'(t)|\, dt =$$
$$\int_0^1 \left\langle 3t + t^2, t^3, t^6 \right\rangle \bullet \left\langle 1, 2t, 3t^2 \right\rangle\, dt = \int_0^1 3t + t^2 + 2t^4 + 3t^8\, dt =$$
$$3/2 + 1/3 + 2/5 + 3/7 = 559/210 \approx 2.6619047619.$$

To compute flux we need to figure out what to use for N and dS. To get N let $g(x, y, z) = x^2 + 2y - z$. Then $N = \bigtriangledown g/|\bigtriangledown g|$. Also $dS = |\bigtriangledown g| dA$. Therefore

$$\text{Flux} = \iint_S F \bullet N\, dS = \iint_U F \bullet \bigtriangledown g\, dA =$$
$$\iint_U \left\langle 3x + y, z, z^2 \right\rangle \bullet \langle 2x, 2, -1 \rangle\, dA = \iint_U (3x + y)2x + 2z - z^2 dA =$$
$$\iint_U (3x + y)2x + 2(x^2 + 2y + 3) - (x^2 + 2y + 3)^2\, dA =$$
$$\iint_U 2x^2 + 2xy - 8y - 3 - x^4 - 4x^2 y - 4y^2\, dA =$$
$$\int_0^{2\pi} \int_0^1 (2r^2 \cos^2\theta + 2r^2 \cos\theta \sin\theta - 8r \sin\theta - 3 - r^4 \cos^4\theta - 4r^3 \cos^2\theta \sin\theta - 4r^2 \sin^2\theta)\, r\, dr\, d\theta =$$
$$\int_0^1 (2\pi r^2 + 0 - 0 - 6\pi - r^4 3\pi/4 - 0 - 4\pi)\, r\, dr = \int_0^1 -10\pi r + 2\pi r^3 - (3\pi/4) r^5\, dr =$$
$$-5\pi + 2\pi/3 - 3\pi/5 = -74/15 \approx -4.9333$$

Integrating over Vector Fields: Short Cuts, FTC, Div Thm, Stokes' Thm.

If F is a **conservative** field, that is it there exits a scalar function f such that $F = \bigtriangledown f$, then the **Fundamental Theorem of Line Integrals** says

$$\int_C F \bullet T ds = \int_a^b \bigtriangledown f \bullet r'(t)\, dt = f(r(b)) - f(r(a))$$

where $r(t)$ with $a \leq t \leq b$ is a parametrization of C. This is called **path independence**. If follows that if C is a closed curve (loop) then

$$\oint_C F \bullet T ds = 0.$$

A vector field F is conservative if and only if $\bigtriangledown \times F = 0$.

(We drop the assumption that F is conservative.) If S is a surface with boundary C (a closed curve) then **Stokes' Theorem** says that

$$\oint_C F \bullet T\, ds = \iint_S (\bigtriangledown \times F) \bullet N\, dS.$$

Notice that if we change the surface S with another surface S' that has the same boundary C the result is unchanged. If S is a closed surface then C is a point and we get that the flux of the curl of F is zero though any closed surface.

In the plane Stokes's Theorem is becomes **Green's Theorem**.

If V is a connected region in with boundary S then the **Divergence Theorem** says if F has continuous derivatives in V then

$$\iint_S F \bullet N\, dS = \iiint_V \bigtriangledown \bullet F\, dV$$

These three theorems are generalizations of the Fundamental Theorem of Calculus, namely, we can evaluate $\int_a^b f(x)\, dx$ from only knowing the anti-derivative on the boundary of the interval $[a, b]$; $\int_a^b f(x)\, dx = F(b) - F(a)$ where $F'(x) = f(x)$. They can greatly simplify many calculations but also, with more experience, yield deep physical insights.

Example: Let V be the cube $[0, 2]^3$ and let S be its surface with outward normal. Let $F = \langle x^2, y^3, e^z - 1\rangle$. Find the flux of F through S.

Solution. By the divergence theorem

$$\iint_S F \bullet N dS = \iiint_V \bigtriangledown \bullet F dV.$$

Now $\bigtriangledown \bullet F = x^2 + y^3 + e^z - 1$. Thus

$$\text{flux} = \int_0^2 \int_0^2 \int_0^2 x^2 + y^3 + e^z - 1\, dx\, dy\, dz = 80 + 4(e^2 - 1).$$

To do this directly we'd need to integrate over each of the 6 faces separately. Yuck!

Example: Let $F = \langle 3x^2, 2y, 2z\rangle$. Find the work done in pushing a particle along the helix $\langle \cos \pi t, \sin \pi t, t^3\rangle$ from (1,0,0) to (1,0,8).

Solution I. Since $\bigtriangledown \times F = \langle 0, 0, 0\rangle$ there is a potential function f with $\bigtriangledown f = F$. In fact it is easy to see that $f(x, y, z) = x^3 + y^2 + z^2$ works. Thus,

$$\int_{\text{Helix}} F \bullet T\, ds = f(1,0,8) - f(1,0,0) = 64.$$

Solution II. If we wanted to do this directly we would have

$$\int_{Helix} F \bullet T\, ds = \int_0^2 \left\langle 3\cos^2 \pi t, 2\sin \pi t, 2t^3 \right\rangle \bullet \left\langle -\pi \sin \pi t, \pi \cos \pi t, 3t^2 \right\rangle\, dt =$$

$$\int_0^2 -3\pi \cos^2 \pi t \sin \pi t + 2\pi \cos \pi t \sin \pi t + 6t^5\, dt = 0 + 0 + 64.$$

Example. Let G be a vector field with all second partial derivatives continuous. Let S be a closed surface with outward normal. What is the flux of the curl of G out through S?

Solution. Let V be the bounded region in $\mathbb{R}^3$ with boundary S. Then

$$\iint_S \bigtriangledown \times G\, dS = \iiint_V \bigtriangledown \bullet (\bigtriangledown \times G) dV = \iiint 0\, dV = 0.$$

Remember for any twice differentiable vector field G, div curl G = 0.

Example. Let $F = \langle x + y + 3z, 2x - z, x - 2y + z \rangle$. Let S be the surface of a bounded region V in $\mathbb{R}^3$ which has volume 23. Find the flux of F out through S.

Solution. Div F = 1+0+1 = 2. Thus

$$\text{flux} = \iint_S F \bullet N dS = \iiint_V 2dV = 2 \times 23 = 46.$$

Example. Let $F = \langle M, N, P \rangle = \left\langle x^3 y, xy^2, \sin \ln \sqrt{x + yz + 7} \right\rangle$. Let S be the portion of the cone $z = 9 - \sqrt{x^2 + y^2}$ above the xy-plane and below $z = 9$ with outward normal vector. Find the flux of $\bigtriangledown \times F$ through S.

Solution. Let C be the circle of radius 3 in the xy-plane with center at the origin, oriented counterclockwise. Then C is the boundary of S. By Stokes' Theorem

$$\iint_S \bigtriangledown \times F \bullet N dS = \oint_C F \bullet T ds.$$

Let S^* be the disk of radius 3 in the xy-plane with center at the origin with upward normal vector. Notice the boundary of S^* is just C. Thus by Stokes' Theorem (or Green's Theorem)

$$\oint_C F \bullet T ds = \iint_{S^*} \bigtriangledown \times F \bullet \langle 0, 0, 1 \rangle\, dA =$$

$$\iint_{S^*} M_y - N_x dA = \iint_{S^*} x^3 - y^2 dA =$$

$$\int_0^{2\pi} \int_0^3 (r^3 \cos^3 \theta - r^2 \sin^2 \theta)\, r\, dr\, d\theta = \int_0^3 0 - r^3 \pi\, dr = -81\pi/4.$$

Some Properties of Vector Fields

Let f be a scalar function. If all second partial derivatives are continuous then

$$\bigtriangledown \times (\bigtriangledown f) = \mathbf{0}$$

Let F be a vector field with continuous second partial derivatives. Then

$$\bigtriangledown \bullet (\bigtriangledown \times F) = 0.$$

As we stated above if $\bigtriangledown \times F = \mathbf{0}$, that is F is conservative, then there exists a scalar function f such that $F = \bigtriangledown f$. In many applications f is the potential energy (although physics books will let $U = -f$ and write $F = -\bigtriangledown U$). We developed a method for finding f.

If G is a vector field with $\bigtriangledown \bullet G = 0$, that if G is divergence free, then there exists another vector field F such that $G = \bigtriangledown \times F$. In this case F is called a vector potential field for G. There is a method for finding F, but we have not covered this.

Example. A static electric force field is conservative and thus has a scalar potential function. This is not true for a magnetic field, but it turns out they are divergence free and so have vector potential field.