Lines and Planes[1]

1 Lines in the Plane

Every line of points L in $\mathbb{R}^2$ can be expressed as the solution set for an equation of the form $Ax + By = C$. Will we call this the *ABC form*. Recall that the *slope-intercept* form is $y = ax + b$. However, vertical lines cannot be expressed in slope-intercept form, but they can be express in ABC form: The vertical line $x = 5$ corresponds to, $A = 1$, $B = 0$, and $C = 5$.

The ABC form for a line L is not unique for if we multiply both sides by any nonzero number the solution set is unchanged. Any line L can also be expressed by a pair of *parametric equations*, by which we mean:

$$\begin{aligned} x(t) &= at + b \\ y(t) &= ct + d \end{aligned}$$

for suitable constants. These can be rewritten in *vector form* as $\langle x, y\rangle = \langle a, c\rangle\, t + \langle b, d\rangle$. The vectors $\langle a, c\rangle$ and $\langle b, d\rangle$ have a nice geometric/physical interpretation.

Regard t as time. One can imagine a particle moving along L in accordance with the given parametric equations. We let $\mathbf{p}(t) = \langle x(t), y(t)\rangle$ and call it the *position vector*. Then $\mathbf{p}(0) = \langle b, d\rangle$ is the *initial position*. Notice,

$$\frac{d\mathbf{p}}{dt} = \langle x'(t), y'(t)\rangle = \langle a, c\rangle$$

(The derivative of a vector of functions is just the vector given by taking the derivative of each component, but we will not need this in this course.) Thus, we call $\mathbf{v} = \langle a, c\rangle$ the *velocity vector*. It is parallel to L. It is customary to place its base point on L. See Figure 1(left side).

We now give a geometric interpretation for the ABC form of an equation of a line. First, suppose $C = 0$; this just means the line L goes through the origin. Let $\mathbf{n} = \langle A, B\rangle$, and again set $\mathbf{p} = \langle x, y\rangle$. Then we have $\mathbf{n} \bullet \mathbf{p} = 0$. That is the vectors $\mathbf{n}$ and $\mathbf{p}$ are at right angles to each other. Thus, the line L for $Ax+By = 0$ is the set of all points (x, y) such that $\langle x, y\rangle$ is perpendicular to $\langle A, B\rangle$.

Now we consider the general case: $Ax + By = C$. Pick some particular point on the line and call it (x_0, y_0). Then $C = Ax_0 + By_0$. Therefore, for any point (x, y) on L we have $Ax + By = Ax_0 + By_0$. We can rewrite this as

$$\begin{aligned} Ax - Ax_0 + By - By_0 &= 0 \\ A(x - x_0) + B(y - y_0) &= 0 \\ \langle A, B\rangle \bullet \langle x - x_0, y - y_0\rangle &= 0 \\ \mathbf{n} \bullet (\langle x, y\rangle - \langle x_0, y_0\rangle) &= 0 \\ \mathbf{n} \bullet (\mathbf{p} - \mathbf{p}_0) &= 0 \end{aligned}$$

In the last line we have let $\mathbf{p}_0 = \langle x_0, y_0\rangle$. The vector $\mathbf{p} - \mathbf{p}_0$ can be thought of as lying in L with its tail at (x_0, y_0) and its head at (x, y).

Thus, L is the unique line perpendicular to the vector $\mathbf{n} = \langle A, B\rangle$ that passes through (x_0, y_0). See Figure 1(right side). The vector $\mathbf{n}$ is called a *normal vector* for the line L. Given a vector to use as normal vector and a point we can easily find an equation for the corresponding line.

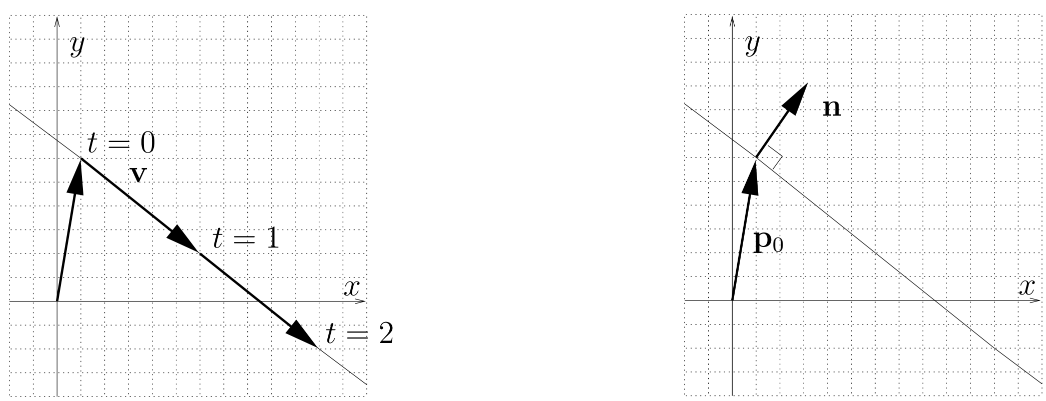

Figure 1: Left: A parametric line. Right: Normal vector to a line.

Problem 1. Consider the line that is determined by the parametric equations $x(t) = 3t - 2$ and $y(t) = -t + 7$. Find an equation for the line in ABC form.

Problem 2. Consider the line determined by $4x - 7y = 2$. Find a pair of parametric equations for this line.

Note: Problems 1 and 2 have many correct answers.

2 Lines and Planes in 3-space

The three dimensional set $\mathbb{R}^3$ is the set of all triples (x, y, z) where x, y, and z are real numbers. Such a triple is called the xyz-coordinates of a point. These are also called *rectilinear coordinates*. The set $\{(x, 0, 0) \,|\, x \in \mathbb{R}\}$ is the x-axis. The y and z axes are defined similarly. They are clearly lines. The set $\{(x, y, 0) \,|\, x \in \mathbb{R}\,,\ y \in \mathbb{R}\}$ is the xy-plane. The yz and xz planes are defined similarly. Visualizing structures in three dimensions takes practice.

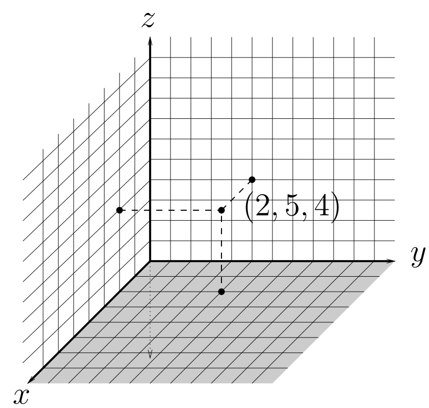

Figure 2: Three dimensional space: $\mathbb{R}^3$

Any line L in $\mathbb{R}^3$ can expressed parametrically in the form:

$$\begin{aligned} x(t) &= at + b \\ y(t) &= ct + d \\ z(t) &= et + f \end{aligned}$$

or, in vector form, $\langle x, y, z\rangle = \langle a, c, e\rangle\, t + \langle b, d, f\rangle$. As with lines in $\mathbb{R}^2$ it is useful to think of $\langle a, c, e\rangle$ as a velocity vector and $\langle b, d, f\rangle$ as the position at $t = 0$.

However, there is no way to express a line in $\mathbb{R}^3$ as a single equation in the three variables.

Now we move on to planes in $\mathbb{R}^3$. Consider the solution set of an equation of the form $Ax + By + Cz = D$. If $A = B = C = D = 0$, the solution set is all of $\mathbb{R}^3$. If $A = B = C = 0$ but $D \neq 0$ the solution set is empty. These two examples should be thought of as degenerate cases. In all other cases the solution set to an equation of the form $Ax + By + Cz = D$ will be a plane in

$\mathbb{R}^3$. We will call this form the *ABCD form*. Parametric equations for planes are described in the next section.

Example 1. Convince yourself of the following:

- If $A = B = D = 0$ and $C \neq 0$ then $Ax + By + Cz = D$ is the xy-plane.
- If $A = C = D = 0$ and $B \neq 0$ then $Ax + By + Cz = D$ is the xz-plane.
- If $B = C = D = 0$ and $A \neq 0$ then $Ax + By + Cz = D$ is the yz-plane.

Example 2. Graph the plane P that is the solution set to $2x + 3y + 4z = 12$ in $\mathbb{R}^3$.

Solution. Will we find the intercepts with each of the three coordinate axes. Let $y = z = 0$. Then $x = 6$. Hence the point $(6, 0, 0)$ in on P. Next let $x = z = 0$. Then $y = 4$. Hence the point $(0, 4, 0)$ is on P. Finally let $x = y = 0$. Then $z = 3$. Hence $(0, 0, 3)$ is on P. In Figure 3 we plot these points and connect them with line segments to help visualize the plane P. □

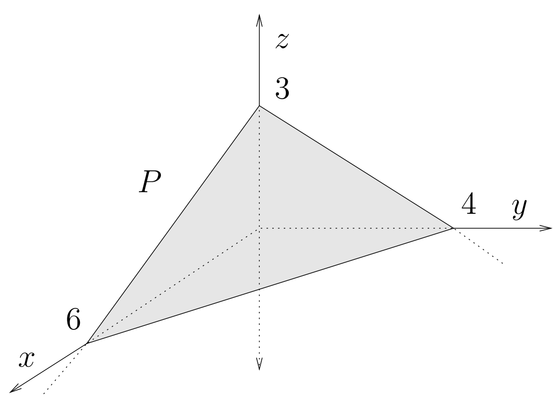

Figure 3: The plane given by $2x + 3y + 4z = 12$ in $\mathbb{R}^3$.

Next we will give a geometric interpretation of an equation for a plane P in ABCD form. First we consider the case where $D = 0$. Let $\mathbf{n} = \langle A, B, C\rangle$ and $\mathbf{p} = \langle x, y, z\rangle$. Then the equation $Ax + By + Cz = 0$ becomes $\mathbf{n} \bullet \mathbf{p} = 0$. Thus, the solution set is the plane P, passing through the origin of $\mathbb{R}^3$ whose points, when regarded as vectors, are perpendicular to $\mathbf{n}$.

We return to the general case: $Ax+By+Cz=D$. Let $\mathbf{p}_0=\langle x_0,y_0,z_0\rangle$ be some fixed point that satisfies the given equation. We leave it to the reader to show that

$$\mathbf{n} \bullet (\mathbf{p}-\mathbf{p}_0)=0.$$

Thus, the solution set of $Ax+By+Cz=D$ is the unique plane passing through $\mathbf{p}_0$ and perpendicular to $\mathbf{n}=\langle A,B,C\rangle$.

Example 3. We reconsider the plane P given by $2x+3y+4z=12$ in Example 2. Let $\mathbf{n}=\langle 2,3,4\rangle$. Pick two points on P, say $\mathbf{p}_1=(2,2,1/2)$ and $\mathbf{p}_2=(4,0,1)$. Regard them as vectors and let $\mathbf{v}=\mathbf{p}_2-\mathbf{p}_1=\langle 2,-2,1/2\rangle$. Then $\mathbf{v} \bullet \mathbf{n}=2\cdot 2+(-2)\cdot 3+(1/2)\cdot 4=4-6+2=0$ as expected.

Example 4. Let P_1 be the plane given by $2x+3y-z=4$ and let P_2 be the plane given by $x+y+z=1$. Find parametric equations for the line $L=P_1\cap P_2$, then rewrite them in vector form.

Solution.

$$\left.\begin{array}{rcl} 2x+3y-z &=& 4 \\ x+y+z &=& 1 \end{array}\right\} \implies y-3z=2.$$

Let $z=t$. Then $y=3t+2$ and $x=1-y-z=-4t-1$. Thus,

$$\begin{array}{rcl} x(t) &=& -4t-1 \\ y(t) &=& 3t+2 \\ z(t) &=& t \end{array}$$

are parametric equations for the line L. Lastly, we can rewrite these in vector form. $\langle x,y,z\rangle=\langle -4,3,1\rangle\, t+\langle -1,2,0\rangle$. □

Example 5. Find an equation for the plane passing through the three points $(1,1,1)$, $(1,2,3)$, and $(2,-1,0)$.

Solution 1. We know an equation for this plane can be expressed in the form

$$\mathbf{n} \bullet (\mathbf{p}-\mathbf{p}_0)=0.$$

For $\mathbf{p}_0$ we can use any point on the plane; let $\mathbf{p}_0=\langle 1,1,1\rangle$. Let $\mathbf{v}_1$ be the vector based at $\mathbf{p}_0$ with the other end at $(1,2,3)$. Let $\mathbf{v}_2$ be the vector based at $\mathbf{p}_0$ with the other end at $(2,-1,0)$. Then we define $\mathbf{n}=\mathbf{v}_1\times\mathbf{v}_2$. It will be perpendicular to our plane.

$$\mathbf{v}_1=\langle 1,2,3\rangle-\langle 1,1,1\rangle=\langle 0,1,2\rangle\,.$$

$$\mathbf{v}_2 = \langle 2, -1, 0 \rangle - \langle 1, 1, 1 \rangle = \langle 1, -2, -1 \rangle .$$

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 3, 2, -1 \rangle .$$

Thus we have,

$$\langle 3, 2, -1 \rangle \bullet \langle x - 1, y - 1, z - 1 \rangle = 0.$$

This simplifies to

$$3x + 2y - z = 4.$$

□

Solution 2. We have three conditions and these give us three equations in four unknowns.

$$\left.\begin{array}{rcrcrcr} A & + & B & + & C & = & D \\ A & + & 2B & + & 3C & = & D \\ 2A & - & B & & & = & D \end{array}\right\} \Longrightarrow$$

$$\left.\begin{array}{rcrcrcr} A & + & B & + & C & = & D \\ & & B & + & 2C & = & 0 \\ & & -3B & - & 2C & = & -D \end{array}\right\} \Longrightarrow$$

$$\left.\begin{array}{rcrcrcr} A & + & B & + & C & = & D \\ & & B & + & 2C & = & 0 \\ & & & & C & = & -D/4 \end{array}\right\} \Longrightarrow$$

$$\left.\begin{array}{rcrcrr} A & + & B & & = & 5D/4 \\ & & B & & = & D/2 \\ & & & C & = & -D/4 \end{array}\right\} \Longrightarrow \begin{array}{rcr} A & = & 3D/4 \\ B & = & D/2 \\ C & = & -D/4 \end{array}$$

Any nonzero value of D will do. Let $D = 4$. Then $3x + 2y - z = 4$ is an equation for our plane. □

Discussion. The cross product method was probably easier. The reason for doing the systems of equations method is that this method will generalize to higher dimensions, whereas the cross product trick only works in dimension three.

Problem 1. Consider the three points $(1, 1, 1)$, $(2, 0, 2)$, and $(4, -2, 4)$. Show that they do not determine a unique plane because they lie on the same line. Find an equation for this line; write it in vector form.

Problem 2. Let P be the plane given by $x + 2y - 3z = 1$. Let L_{xy} be the intersection of P with the xy-plane. Find an equation for this line ABC form and slope-intercept form.

Problem 3. Graph, separately, each of the planes determined by these three equations: $2x + 2y - 3z = 1$, $x + 2y + 4z = -1$, and $3x - 2y - 2z = 7$.

Problem 4. Find the point of intersection of the three planes determined by these three equations: $2x+2y-3z = 1$, $x+2y+4z = -1$, and $3x-2y-2z = 7$.

Problem 5. Show that the two planes determined by $2x + 2y - 3z = 1$ and $4x + 4y - 6z = 0$ do not intersect and are thus parallel.

Problem 6. Let P be the plane given by $2x + 3y - 2z = 1$. Let L be the line given by $\langle x, y, z\rangle = \langle 1, 1, 1\rangle t + \langle 1, 0, 1\rangle$. Find the point where they meet.

Problem 7. Show that these four points lie in the same plane: $(1, 1, -1)$, $(-1, 0, 0)$, $(-1, 1, \frac{1}{2})$, and $(1, -1, 0)$. Find an equation for this plane.

3 Parametric Equation for a Plane

There is another form for equations of planes in $\mathbb{R}^3$ that is the analog of the parametric form for equations of a line. The difference is we will need two parameters, r and s, instead of one. Of course, the time metaphor is no longer useful.

Let P be a plane given by $Ax+By+Cz = D$. Assume that $C \neq 0$. Then we can solve for z and get $z = D/C - A/Cx - B/Cy$. (If $C = 0$ solve for x or y instead.) Think of z as the height above the xy-plane. Now let $x = r$ and $y = s$, and think of r and s as free parameters. We can now write

$$\begin{aligned}\langle x, y, z\rangle &= \langle r, s, D/C - A/Cr - B/Cs\rangle \\ &= \langle 0, 0, D/C\rangle + r\,\langle 1, 0, -A/C\rangle + s\,\langle 0, 1, -B/C\rangle\end{aligned}$$

This equation is far from unique. We can start with any point $(x_0, y_0, z_0) \in P$, regard it as a vector $\mathbf{p}_0 = \langle x_0, y_0, z_0\rangle$ and add multiplies of $\langle 1, 0, -A/C\rangle$ and $\langle 0, 1, -B/C\rangle$ to it and stay in the plane. Furthermore, if we let $\mathbf{v}_1$ and $\mathbf{v}_2$ be nonzero multiplies of $\langle 1, 0, -A/C\rangle$ and $\langle 0, 1, -B/C\rangle$, respectively then

$$\mathbf{p} = \mathbf{p}_0 + r\mathbf{v}_1 + s\mathbf{v}_2$$

gives the same plane P. Indeed, we could use any pair of vectors in P with tails at $\mathbf{p}_0$ as long as they point in different directions.

We will use this formulation to place a coordinate system on P. Take a point (x_0, y_0, z_0) on P and call it the origin of P. Then any point on P can be gotten to by adding multiples of $\mathbf{v}_1$ and $\mathbf{v}_2$ to $\mathbf{p}_0$. Thus, for any point on P we can think of it as having coordinates (r, s). See Figure 4.

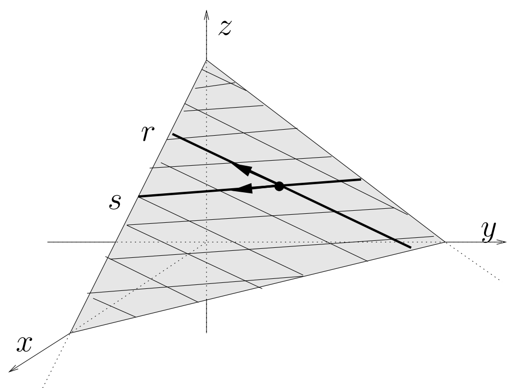

Figure 4: Coordinates for a plane: The dark lines are the r and s-axes

Example 1. Define a plane P by

$$\langle x, y, z\rangle = \langle 1, 2, 3\rangle + r\,\langle 1, 1, 0\rangle + s\,\langle 0, 1, 1\rangle$$

Show that the point (0,2,4) is on P and find its rs-coordinates.

Solution. We have three equations and two unknowns.

$$\left.\begin{array}{l} 0 = 1 + 1r + 0s \\ 2 = 2 + 1r + 1s \\ 4 = 3 + 0r + 1s \end{array}\right\} \quad \Longrightarrow \quad \begin{array}{c} r = -1 \\ s = 1 \end{array}$$

Thus, $(0, 2, 4) \in P$ and it has rs-coordinates $(-1, 1)$ relative to the given parametric equation. □

Problem 1. Using the same plane P in Example 1, find the rs-coordinates of $(3, 3, 2)$.

Problem 2. Show that the point $(1, 2, -1)$ is not on the plane P of Example 1.

Problem 3 (Hard)**.** The equation $2r + 3s = 1$ determines a line L in the plane P of Example 1, using rs-coordinates. Find a parametric equation for L in xyz-coordinates.

4 Summary

I. For lines in $\mathbb{R}^2$ we have studied three forms, the **ABC form**, **parametric form**, and **vector form**.

$$Ax + By = C \qquad \begin{array}{rcl} x(t) & = & at + b \\ y(t) & = & ct + d \end{array} \qquad \langle x, y \rangle = \langle a, c \rangle\, t + \langle b, d \rangle$$

II. For lines in $\mathbb{R}^3$ we do not have an analog of the ABC form, but we can express any line in $\mathbb{R}^3$ in **parametric** and **vector form**.

$$\begin{array}{rcl} x(t) & = & at + b \\ y(t) & = & ct + d \\ z(t) & = & et + f \end{array} \qquad \langle x, y, z \rangle = \langle a, c, e \rangle\, t + \langle b, d, f \rangle$$

III. For planes in $\mathbb{R}^3$ we studied the **ABCD form**, the **parametric form**, and the **vector form**.

$$Ax + By + Cz = D \qquad \begin{array}{rcl} x(r,s) & = & ar + bs + c \\ y(r,s) & = & dr + es + f \\ z(r,s) & = & gr + hs + i \end{array}$$

$$\langle x, y, z \rangle = \langle a, d, g \rangle\, r + \langle b, e, h \rangle\, s + \langle c, f, i \rangle$$

For all three cases the you should be able to convert equations of one form into the others. You should begin to wonder what sorts of linear structures exist in $\mathbb{R}^n$ for $n > 3$.