# Curvature[1]

The curvature of a circle of radius R is defined to be $\kappa = 1/R$. It is a measure of how sharply we are turning as we travel along the circle at a steady speed. We want to generalize this idea to define the curvature of any smooth curve. To this end we study the case of a circle more carefully.

Below is a circle of radius R. As we move along an arc of length s the radial lines sweep out a central angle θ. Consider the two unit tangent vectors T_1 and T_2.
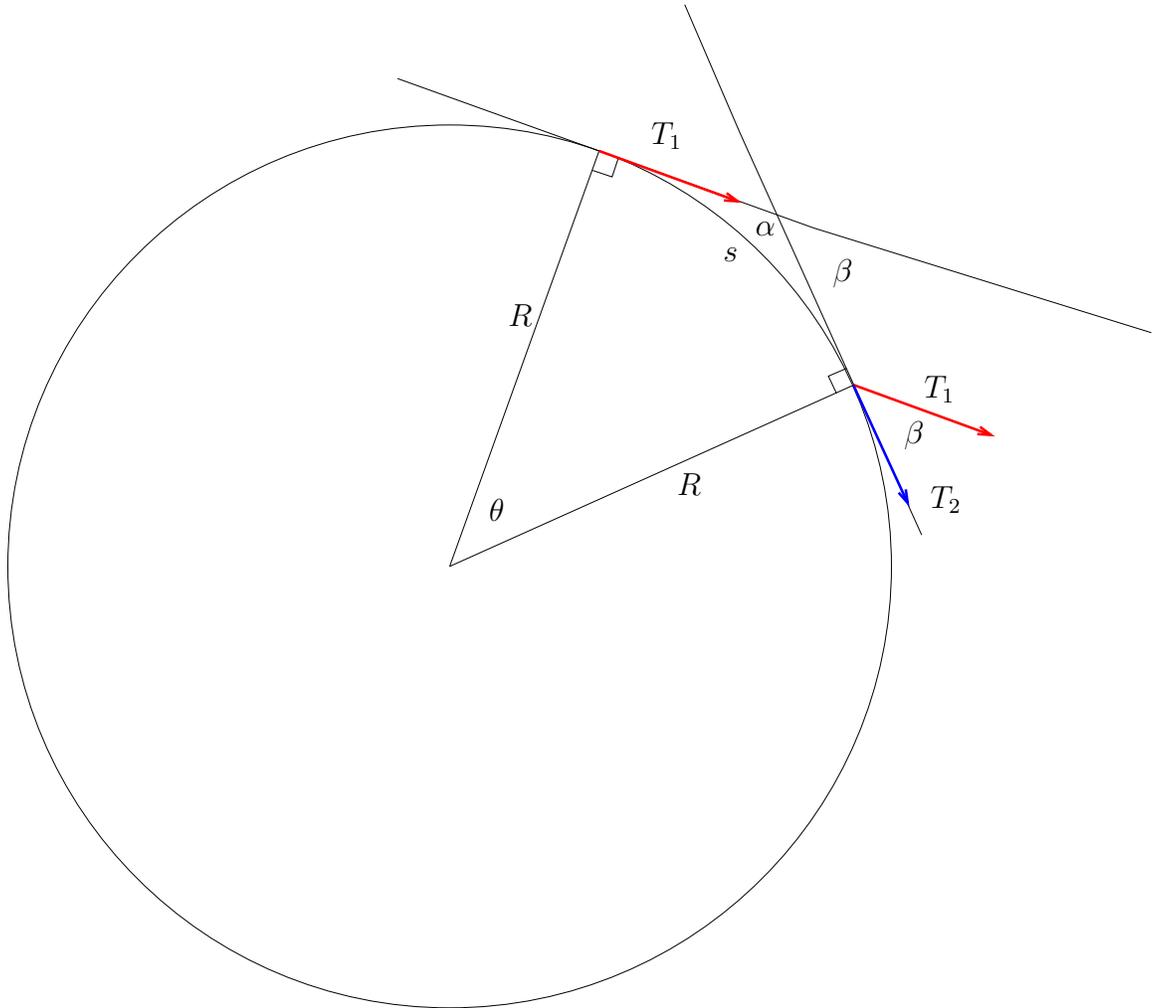
We claim that the angle θ formed between the radial lines as we move s units along the circle is the same as the angle β between the

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unit tangent vectors T_1 and T_2. To see this recall that the interior angles of a quadrilateral add to 2π radians. Since the two angles formed by the two radii and their respective tangent lines are $\pi/2$ radians,

$$\theta + \alpha = \pi.$$

But since we also have $\alpha + \beta = \pi$ it must be that $\beta = \theta$.

Now imagine that the arc length traveled Δs is very small and hence the angle formed at the circle's center $\Delta\theta$ is also small. Let $\Delta T = T_2 - T_1$.

The figure below illustrates that $|\Delta T| \approx \Delta\theta$. Since, $|T_1| = |T_2| = 1$ we are on the unit circle. Thus, the radian measure of the angle $\Delta\theta$ is the length of the arc formed.
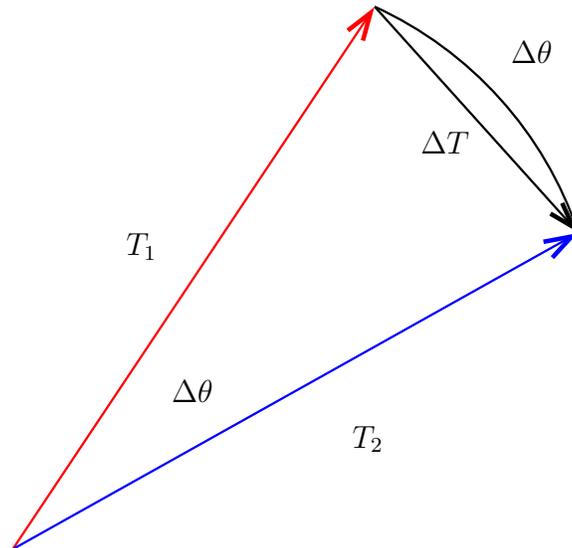
Taking the limit as $\Delta\theta \to 0$ it can be shown that $\dfrac{|\Delta T|}{\Delta\theta} \to 1$.
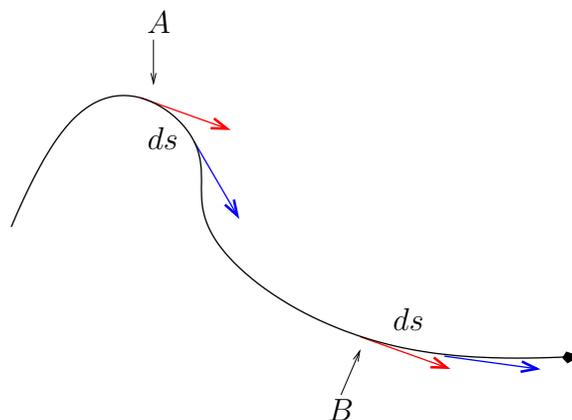
Going back to the original circle of radius R we have $\Delta s = R\Delta\theta$. Hence, as $\Delta s \to 0$ we have

$$\left|\frac{\Delta T}{\Delta s}\right| = \frac{1}{R}\left|\frac{\Delta T}{\Delta\theta}\right| \to \frac{1}{R} = \kappa.$$

This motivates the general definition of the curvature of a smooth curve to be

$$\kappa = \left|\frac{dT}{ds}\right|.$$

The figure below illustrates this for a general curve. Notice the curvature at the point A is quite a bit larger than the curvature at point B.

Computations.

This definition is however hard to work with. For a curve given by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ we would need to reparameterize it in terms of arc length, compute the unit tangent vector, take its derivative and then find the magnitude. The textbook derives several handy formals for finding the curvature in different contexts.

For a curve in $\mathbb{R}^3$ given by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ we have

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

If the curve is in $\mathbb{R}^2$ then $z(t) = 0$ and this reduces to

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{\left(\sqrt{[x'(t)]^2 + [y'(t)]^2}\right)^3}.$$

If the curve is in $\mathbb{R}^2$ and is given by $y = f(x)$, we let $x = t$, and then $\mathbf{r}(t) = \mathbf{r}(x) = \langle x, f(x), 0 \rangle$. Now

$$\kappa(x) = \frac{|f''(x)|}{\left(\sqrt{1 + [f'(x)]^2}\right)^3}.$$

The Osculating circle.

Given a smooth curve $\mathbf{r}(t)$ we define the *osculating circle* at $t = t_0$ to be the circle in the osculating plane that has radius $1/\kappa(t_0)$, meets the curve at $\mathbf{r}(t_0)$ and such that the vector from $\mathbf{r}(t_0)$ to its center points in the same direction as the principal unit normal vector.

Example 1. To see this we give a elementary example. Let's find the osculating circle for $y = x^2$ at $x = 1$. We can take $x = t$ and $\mathbf{r}(x) = \langle x, x^2, 0 \rangle$. Then $\mathbf{r}(1) = \langle 1, 1, 0 \rangle$, $\kappa(1) = \frac{2}{5\sqrt{5}}$ and the osculating plane is the xy-plane. The slope of the tangent line to $y = x^2$ at $x = 1$ is 2. Thus, the principal unit normal vector is $\mathbf{N}(1) = \left\langle -2/\sqrt{5}, 1/\sqrt{5} \right\rangle$. You can derive this from the definition,

$$\mathbf{N}(1) = \mathbf{T}'(1)/|\mathbf{T}'(1)|,$$

or by elementary trigonometry.

Now the center of the osculating circle is $\langle 1, 1 \rangle + (1/\kappa(1))\mathbf{N}(1) = \langle -4, 3.5 \rangle$. Check this. The figure below shows the parabola in red, the tangent line in pink, the osculating circle in green and its radius in blue.
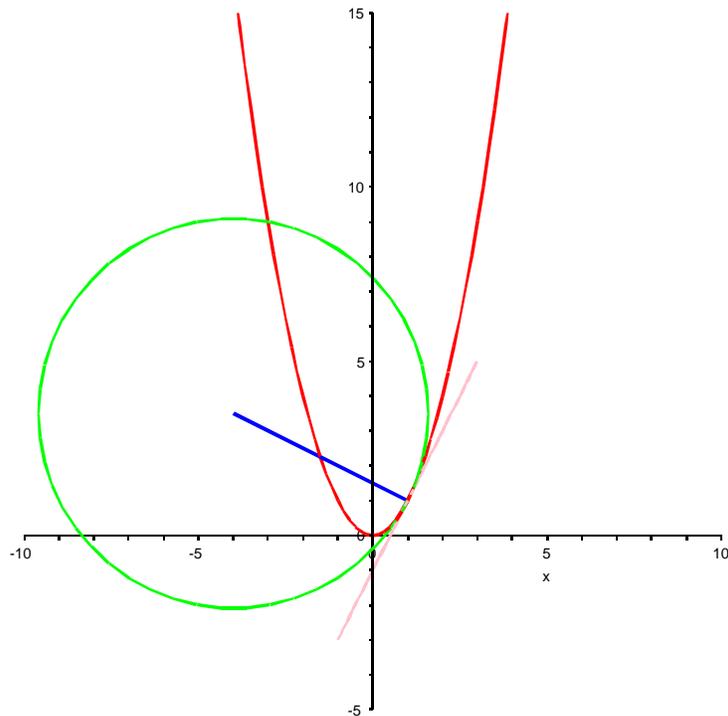
More Examples.

2. Find the curvature of $\mathbf{r}(t) = \langle t \sin t, \cos t, t^2 \rangle$ when $t = \pi$.

Solution. We compute $\mathbf{r}'(t) = \langle \sin t + t \cos t, -\sin t, 2t \rangle$ and $\mathbf{r}''(t) = \langle 2 \cos t - t \sin t, -\cos t, 2 \rangle$.

Therefore, $\mathbf{r}'(\pi) = \langle -\pi, 0, 2\pi \rangle$, $|\mathbf{r}'(\pi)| = \sqrt{5}\,\pi$ and $\mathbf{r}''(\pi) = \langle -2, 1, 2 \rangle$.

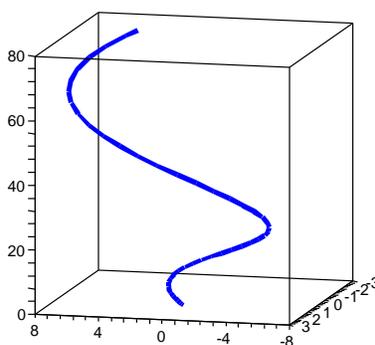Next we find the cross product and then its magnitude.

$$\mathbf{r}'(\pi) \times \mathbf{r}''(\pi) = \langle -\pi, 0, 2\pi \rangle \times \langle -2, 1, 2 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\pi & 0 & 2\pi \\ -2 & 1 & 2 \end{vmatrix} = \langle -2\pi, -2\pi, -\pi \rangle.$$

$|\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)| = 3\pi.$

Putting this all together we have

$$\kappa(\pi) = \frac{|\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)|}{|\mathbf{r}'(\pi)|^3} = \frac{3\pi}{(\sqrt{5}\,\pi)^3} = \frac{3}{5\sqrt{5}\,\pi^2} = \frac{3\sqrt{5}}{25\pi^2} \approx 0.027187327.$$

The curve is shown below.

3. Repeat for $t = 0$.

Solution. But $\mathbf{r}'(0) = \langle 0, 0, 0 \rangle$. So, we would get $0/0$. The problem is an artifact of the parameterization. But, we can compute $\kappa(t)$ and then take the limit as $t \to 0$. However, the calculations are extremely messy. So, I used the program Maple. The commands are below.

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> k := Curvature( t -> <t * sin(t), cos(t), t^2 > ):
> limit(k(t), t=0);
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The result is $2\sqrt{2}/27 \approx 0.1047565601$.

4. Consider the ellipse given by $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$. Find the curvature at each point. For extra fun, create a computer animation of the osculating circle.
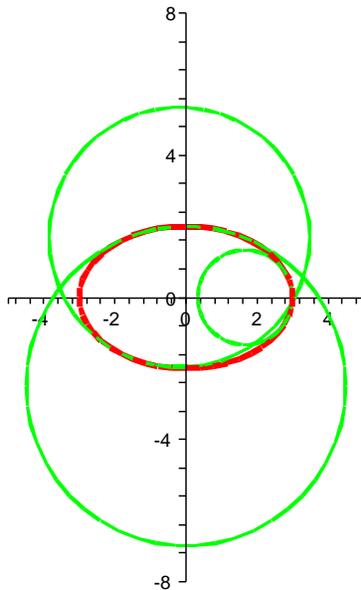
Solution. We will work with a parametric equation for this ellipse.
$$\mathbf{r}(t) = \langle 3\cos t, 2\sin t \rangle,$$
for $t \in [0, 2\pi]$. Then
$$\kappa(t) = \frac{|6\sin^2 t + 6\cos^2 t|}{\left(\sqrt{9\sin^2 t + 4\cos^2 t}\right)^3} = 6(4 + 5\sin^2 t)^{-\frac{3}{2}}.$$
Below is a plot of the ellipse with three different osculating circles.

The animation is given in a separate page.