Vectors. $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$, $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$, $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$, $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$.
The directional derivative of $f(x,y,z)$ in the direction of a unit vector $\mathbf{u}$ is $D_{\mathbf{u}}f = \bigtriangledown f\cdot\mathbf{u}$.

Curves. If $y = f(x)$ the **arc length** of the graph from $x = a$ to $x = b$ is $L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$.
If a curve C is given by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \le t \le b$, then the arc length is

$$L = \int_C ds = \int_a^b \left|\frac{d\mathbf{r}}{dt}\right| dt = \int_a^b \sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt.$$

If $f(x,y,z)$ is a density function the line integral of f along C is

$$\int_C f\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt.$$

If $\mathbf{F}(x,y,z)$ is a vector field the **work** done by $\mathbf{F}$ in moving an object along C is

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}\cdot\frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|}\left|\frac{d\mathbf{r}}{dt}\right| dt = \int_a^b \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt.$$

If $\mathbf{F} = \langle P, Q, R\rangle$ some people write this as $\int \mathbf{F}\cdot d\mathbf{r} = \int_c P\,dx + Q\,dy + R\,dz$. If C is a closed loop this is also called the **circulation**.

If $\mathbf{F}$ is conservative, that is if $\bigtriangledown\times\mathbf{F} = \mathbf{0}$, then there is a scalar function f such that $\mathbf{F} = \bigtriangledown f$ and by the **Fundamental Theorem of Line Integrals**

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = f(\text{end point}) - f(\text{start point}).$$

Further, if C is a closed loop we have $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = 0$.

The unit tangent, unit normal and unit binormal vectors to C are
$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$, $\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t)$ and the curvature is $\kappa(t) = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.
Let $\mathbf{v}(t) = \mathbf{r}'(t)$, $v = |\mathbf{v}(t)|$ and $\mathbf{a}(t) = \mathbf{r}''(t)$. Then $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, where

$$a_T = v' = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} \text{ and } a_N = \kappa v^2 = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}.$$

Surfaces. Let $z = f(x,y)$ be a smooth surface S with $f_x(a,b) = f_y(a,b) = 0$. Let $D = f_{xx}f_{yy} - f_{xy}^2$, all evaluated at $x = a$, $y = b$. Then $D > 0$ & $f_{xx} > 0$ implies $f(a,b)$ is a local minimum, $D > 0$ & $f_{xx} < 0$ implies $f(a,b)$ is a local maximum, and $D < 0$ implies $f(a,b)$ is a saddle point.

The surface area of S, given by$z = f(x,y)$, over some domain D in the xy-plane is

$$\iint_S dS = \iint_D \sqrt{1+[f_x]^2+[f_y]^2}\,dA,$$

where dA might be $dx\,dy$, or $dy\,dx$; or you can convert to polar: $r\,dr\,d\theta$ or $r\,d\theta\,dr$.

If a surface S is give by a parameterization $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ where (u,v) is over some domain D in uv-space, then the surface area is

$$\iint_S dS = \iint_D |\mathbf{r}_u\times\mathbf{r}_v|\,dA,$$

where $dA = du\,dv$, $dv\,du$, or you may wish to convert to to some other coordinates.

A special case we did was when the surface is a sphere of radius R centered at the origin. Then

$$|\mathbf{r}_\theta\times\mathbf{r}_\phi| = R^2\sin\phi,$$

and $dA = d\theta\,d\phi$ or $d\phi\,d\theta$.

If $g(x,y,z)$ is a density function then the surface integral of g over S is

$$\iint_S g\,dS = \iint_D g(x,y,f(x,y))\sqrt{1+[f_x]^2+[f_y]^2}\,dA,$$

when S is determined by $z = f(x,y)$. In the other cases, modify dS as above and compute g in terms of whatever variables you are using.

If $\mathbf{F}$ is a vector field the flux of $\mathbf{F}$ through S is given by $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{N}\,dS$.
Here $\mathbf{N}$ is a unit normal vector to the surface S. If S is given by $z = f(x,y)$ then

$$\mathbf{N} = \pm\left\langle f_x, f_y, -1\right\rangle/\sqrt{1+(f_x)^2+(f_y)^2}.$$

In the case that the surface is given as a level surface, $h(x,y,z) =$ a constant then

$$\mathbf{N} = \pm\frac{\bigtriangledown h}{|\bigtriangledown h|} \qquad \text{; but } |\bigtriangledown h| \text{ need not equal } |\mathbf{r}_u \times \mathbf{r}_v|\text{; you still have to parameterize } S.$$

When S is the sphere, $x^2+y^2+z^2 = R^2$, you can use $\mathbf{N} = \pm\left\langle x,y,z\right\rangle/R$ and $dS = R^2\sin\phi\,d\theta\,d\phi$.

You compute $\mathbf{F}\cdot\mathbf{N}$ and then do the integration just like a standard surface integral.

Stokes' Theorem. Let S be a surface with boundary C and let $\mathbf{F}$ be a vector field. The circulation around C is the net result of the curl of $\mathbf{F}$ over S. That is

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S (\bigtriangledown\times\mathbf{F})\cdot\mathbf{N}\,dS.$$

Green's Theorem. In the xy-plane, if $\mathbf{F} = \langle A, B\rangle$, then Stokes' Theorem reduces to

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint B_x - A_y,\ dA.$$

The Divergence Theorem. Let S be a surface that encloses a finite region R in $\mathbb{R}^3$. Then the flux out through S is the net result of the divergence of $\mathbf{F}$ through out R. That is

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_R \bigtriangledown\cdot\mathbf{F}\,dV,$$

where dV is $dx\,dy\,dz$, $r\,dr\,d\theta\,dz$, $\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$, or permutations of these, depending on the coordinate system you have chosen.

Moments and Gyrations in the plane. Let D be a region in the plane with density function $\rho(x,y)$. Then

$$m = \iint_D \rho(x,y)\,dA \qquad M_x = \iint_D y\rho(x,y)\,dA \qquad M_y = \iint_D x\rho(x,y)\,dA$$

$$\overline{x} = \frac{M_y}{m} \qquad \overline{y} = \frac{M_x}{m}$$

where m is the mass, M_x is the moment with respect to the x-axis, M_y is the moment with respect to the y-axis, and $(\overline{x},\overline{y})$ is the center of mass. Also,

$$I_x = \iint_D y^2\rho(x,y)\,dA, \qquad I_y = \iint_D x^2\rho(x,y)\,dA, \qquad I_0 = I_x + I_y, \qquad R_g = \sqrt{\frac{I_0}{m}},$$

where I_x is the moment of inertia with respect to the x-axis, $I_y$is the moment of inertia with respect to the y-axis, I_0 is the moment of inertia with respect to the origin, and R_g is the radius of gyration with respect to the origin.

Moments and Gyrations in 3-space. Let R be a region in $\mathbb{R}^3$ and let $\rho(x,y,z)$ be a density function. Then

$$m = \iiint_R \rho(x,y,z)\,dV \qquad M_{yz} = \iiint_R x\rho(x,y,z)\,dV \qquad M_{xz} = \iiint_R y\rho(x,y,z)\,dV \qquad M_{xy} = \iiint_R z\rho(x,y,z)\,dV$$

$$\overline{x} = \frac{M_{yz}}{m} \qquad \overline{y} = \frac{M_{xz}}{m} \qquad \overline{z} = \frac{M_{xy}}{m}$$

where m is the mass, M_{yz} is the moment with respect to the x-axis, M_{xz} is the moment with respect to the y-axis, M_{xy} is the moment with respect to the z-axis and $(\overline{x},\overline{y},\overline{z})$ is the center of mass. Also,

$$I_x = \iiint_R (y^2+z^2)\rho(x,y,z)\,dV \qquad I_y = \iiint_R (x^2+z^2)\rho(x,y,z)\,dV \qquad I_z = \iiint_R (x^2+y^2)\rho(x,y,z)\,dV$$

$$R_g = \sqrt{\frac{I_z}{m}},$$

where I_x is the moment of inertia with respect to the x-axis, I_y is the moment of inertia with respect to the y-axis, I_z is the moment of inertia with respect to the z-axis, and R_g is the radius of gyration with respect to the z-axis.

The kinetic energy from linear motion is $\frac{1}{2}mv^2$. The kinetic energy from rotation is $\frac{1}{2}I\omega^2$, where ω is the angular velocity and I is the moment of inertia about the axis of revolution.