Lagrange Multipliers

Ex Let $f(x,y,z) = 3x + 2y^2 + xz$.
Find the maximum and minimum values of $f(x,y,z)$ when restricted to the unit sphere with center at the origin.

Sol Let $g(x,y,z) = x^2+y^2+z^2$. Then the constraint equation is $g(x,y,z) = 1$.

We need to solve the following system of equations.

$$\left.\begin{array}{c} \nabla f = \lambda \nabla g \\ g = 1 \end{array}\right\} \star$$

$$\nabla f = \langle 3+z,\ 4y,\ x \rangle.$$
$$\nabla g = \langle 2x,\ 2y,\ 2z \rangle.$$

Now $\star$ becomes

$$3+z = 2\lambda x \quad \text{(i)}$$
$$4y = 2\lambda y \quad \text{(ii)}$$
$$x = 2\lambda z \quad \text{(iii)}$$
$$x^2+y^2+z^2 = 1. \quad \text{(iv)}$$

Notice (ii) implies $\lambda = 2$ or $y = 0$.

Suppose $\lambda = 2$. (i) and (iii) become

$$3 + z = 4x$$
$$x = 4z$$

Thus $3 + z = 16z$ so $z = 1/5$, and $x = 4/5$.

Now we find y using (iv).

$$\left(\frac{4}{5}\right)^2 + y^2 + \left(\frac{1}{5}\right)^2 = 1$$

$$y^2 = 1 - \frac{16}{25} - \frac{1}{25} = \frac{8}{25}.$$

Thus $y = \pm \frac{2\sqrt{2}}{5}$.

So we have two points where the normal vectors will be parallel:

$$\boxed{\left(\frac{4}{5}, \frac{2\sqrt{2}}{5}, \frac{1}{5}\right) \text{ and } \left(\frac{4}{5}, \frac{-2\sqrt{2}}{5}, \frac{1}{5}\right)}$$

Now suppose $y=0$. (i) (iii) and (iv) become

$$3+z=2\lambda x \qquad (i')$$
$$x=2\lambda z \qquad (iii')$$
$$x^2+z^2=1. \qquad (iv')$$

I want to isolate λ. That requires divided by x and z. But what if x or z is zero?

Notice if $z=0$, then $x=0$ by (iii'). Then it is impossible to solve (iv'). Thus, $z\neq 0$.

If $x=0$ and $z\neq 0$ then $\lambda=0$ and $z=-3$. But $x=0$, $z=-3$ does not solve (iv'). Thus, $x\neq 0$.

Now we can divide and get

$$\frac{3+z}{x}=2\lambda=\frac{x}{z}$$

Hence
$$3z+z^2=x^2$$
$$3z+2z^2=x^2+z^2=1 \qquad \text{using (iv')}$$
$$2z^2+3z-1=0$$

So,
$$z=\frac{-3\pm\sqrt{17}}{4}.$$

If $z = \frac{-3-\sqrt{17}}{4} \approx -1.78$ we cannot solve

$$x^2 + z^2 = 1.$$

Thus,

$$z = \frac{-3+\sqrt{17}}{4} \approx 0.280776406.$$

is the only candidate. Then

$$x = \pm\sqrt{1-z^2} \approx \pm 0.959773207...$$

We have two more pts to check

$$\left(\frac{1}{2}\sqrt{\frac{-5+3\sqrt{17}}{2}}, 0, \frac{-3+\sqrt{17}}{4}\right)$$

$$\left(-\frac{1}{2}\sqrt{\frac{-5+3\sqrt{17}}{2}}, 0, \frac{-3+\sqrt{17}}{4}\right)$$

We substitute each of these four points into $f(x, y, z)$.

$$f\left(\frac{4}{5}, \frac{2\sqrt{2}}{5}, \frac{1}{5}\right) = \frac{16}{5} = 3.2$$

$$f\left(\frac{4}{5}, \frac{-2\sqrt{2}}{5}, \frac{1}{5}\right) = 3.2$$

$$f\left(\frac{1}{2}\sqrt{\frac{-5+3\sqrt{17}}{2}}, 0, \frac{-3+\sqrt{17}}{4}\right) \approx 3.1488$$

$$f\left(-\frac{1}{2}\sqrt{\frac{-5+3\sqrt{17}}{2}}, 0, \frac{-3+\sqrt{17}}{4}\right) \approx -3.1488.$$