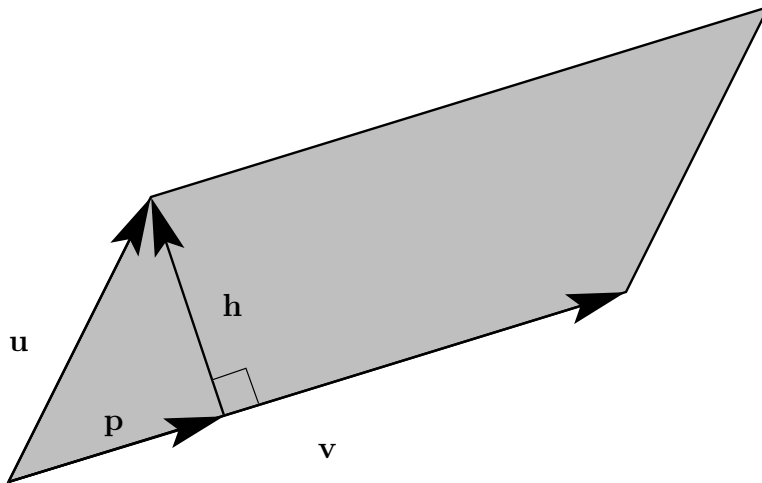


Cross Product Parallelogram Theorem A Direct Proof

In Stewart's calculus textbooks it is proven that the magnitude of the cross product of two nonzero vectors is the area of the parallelogram determined by the two vectors. The proof is correct but not transparent. By that I mean it is not at all obvious how anyone came up with the idea for the result in the first place. Here I'll present a different proof that is geometrically motivated.

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be two vectors based at the origin of \mathbb{R}^3 . In the figure below we can see that the area, A , of the parallelogram is $|\mathbf{h}| |\mathbf{v}|$ where \mathbf{h} is perpendicular to \mathbf{v} . Let \mathbf{p} be the projection of \mathbf{u} onto \mathbf{v} . Then $\mathbf{p} + \mathbf{h} = \mathbf{u}$, so $\mathbf{h} = \mathbf{u} - \mathbf{p}$. Thus, we now have

$$A = |\mathbf{u} - \mathbf{p}| |\mathbf{v}|.$$



We know that $\mathbf{p} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v}$. Thus, $A = \left| \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} \right| |\mathbf{v}|$. It is easier to work with the squares of the magnitudes, so we write

$$A^2 = \left| \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} \right|^2 |\mathbf{v}|^2.$$

Now $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$ and

$$\begin{aligned} \left| \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} \right|^2 &= \left(\mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} \right) \cdot \left(\mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} \right) \\ &= \mathbf{u} \cdot \mathbf{u} - 2 \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\mathbf{v} \cdot \mathbf{v}} + \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\mathbf{v} \cdot \mathbf{v}} = \mathbf{u} \cdot \mathbf{u} - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\mathbf{v} \cdot \mathbf{v}}. \end{aligned}$$

(Write this out for yourself and study it until you understand it.) We can now write

$$A^2 = (\mathbf{u} \bullet \mathbf{u})(\mathbf{v} \bullet \mathbf{v}) - (\mathbf{u} \bullet \mathbf{v})^2.$$

Next we put in the components of \mathbf{u} and \mathbf{v} to get

$$A^2 = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2$$

$$= \begin{pmatrix} u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + \\ u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + \\ u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2 \end{pmatrix} - \begin{pmatrix} u_1^2v_1^2 + u_1v_1u_2v_2 + u_1v_1u_3v_3 + \\ u_2v_2u_1v_1 + u_2^2v_2^2 + u_2v_2u_3v_3 + \\ u_3v_3u_1v_1 + u_3v_3u_2v_2 + u_3^2v_3^2 \end{pmatrix}.$$

Notice that the “diagonal” terms cancel and that the “cross diagonal” terms in the second grouping are identical. Thus,

$$A^2 = [u_1^2v_2^2 - 2u_1v_1u_2v_2 + u_2^2v_1^2] + [u_1^2v_3^2 - 2u_1v_1u_3v_3 + u_3^2v_1^2] + [u_2^2v_3^2 - 2u_3v_3u_2v_2 + u_3^2v_2^2]$$

$$= [u_1v_2 - u_2v_1]^2 + [u_1v_3 - u_3v_1]^2 + [u_2v_3 - u_3v_2]^2 = |\mathbf{u} \times \mathbf{v}|^2.$$

Thus, we have proved the theorem. We note that from simple trigonometry we have $A = |\mathbf{h}| |\mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta$, where θ is the angle going from \mathbf{v} to \mathbf{u} . Therefore we have

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta,$$

while for the dot product we had

$$\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta.$$

Here is a guide to the calculation above.

$$\begin{pmatrix} u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + \\ u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + \\ u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2 \end{pmatrix} - \begin{pmatrix} u_1^2v_1^2 + u_1v_1u_2v_2 + u_1v_1u_3v_3 + \\ u_2v_2u_1v_1 + u_2^2v_2^2 + u_2v_2u_3v_3 + \\ u_3v_3u_1v_1 + u_3v_3u_2v_2 + u_3^2v_3^2 \end{pmatrix}$$

$$[u_1^2v_2^2 - 2u_1v_1u_2v_2 + u_2^2v_1^2] + [u_1^2v_3^2 - 2u_1v_1u_3v_3 + u_3^2v_1^2] + [u_2^2v_3^2 - 2u_3v_3u_2v_2 + u_3^2v_2^2]$$