

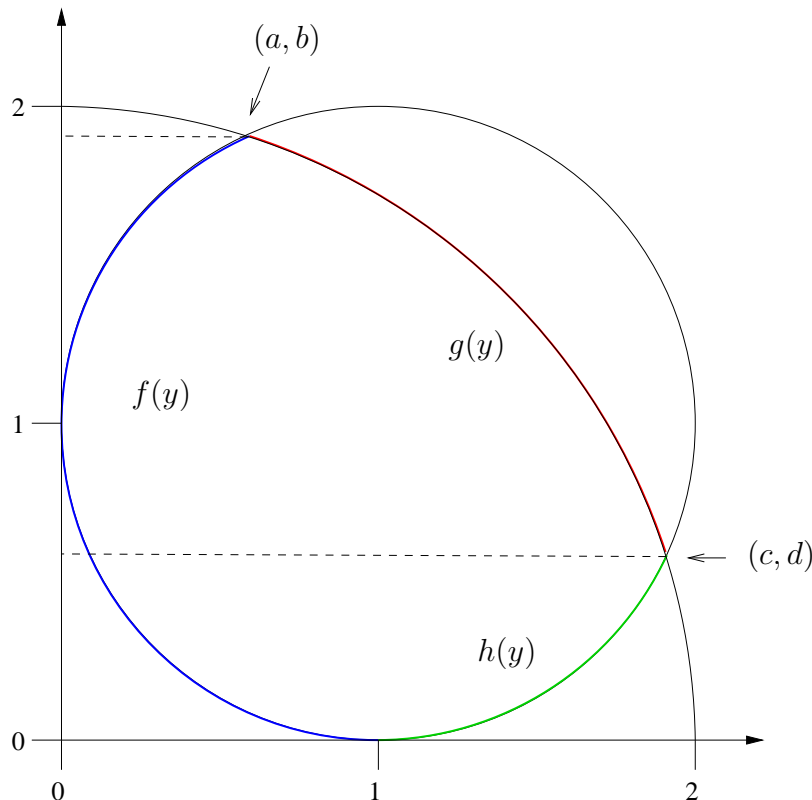
Example of Setting Up A Double Integral for a CAS.

Problem. Setup an integral to find the volume under the graph of $z = 2x + y$ over the region R in the xy -plane defined by

$$R = \{(x, y) \mid x^2 + y^2 \leq 4 \ \& \ (x - 1)^2 + (y - 1)^2 \leq 1\}.$$

Then compute the integral using a computer.

Solution. Below is a graph showing the region R . It is the intersection of two disks, one centered at the origin with radius 2 and the other centered at $(1, 1)$ with radius 1.



The intersection points of the two boundary circles are marked with coordinates (a, b) and (c, d) . We are going to setup two $dx dy$ integrals. To this end we labeled three functions of y in the graph above; $x = f(y)$ is the blue arc, $x = g(y)$ is the red arc, and $x = h(y)$ is the green arc. Then the volume in question is given by

$$V = \int_0^d \int_{f(y)}^{h(y)} 2x + y \, dx \, dy + \int_d^b \int_{f(y)}^{g(y)} 2x + y \, dx \, dy.$$

We solve $x^2 + y^2 = 4$ for x and get $x = \pm\sqrt{4 - y^2}$. The positive branch gives $g(y) = \sqrt{4 - y^2}$.

We solve $(x-1)^2 + (y-1)^2 = 1$ for x and get $x = \pm\sqrt{1 - (y-1)^2} + 1 = 1 \pm \sqrt{2y - y^2}$. The positive branch gives $f(y)$ and the negative branch gives $h(y)$. We record these results below.

$$\begin{aligned}g(y) &= \sqrt{4 - y^2}, \\f(y) &= -\sqrt{2y - y^2}, \\h(y) &= \sqrt{2y - y^2}.\end{aligned}$$

Now we have to find the intersection points (a, b) and (c, d) . For this we need to solve the following system of equations.

$$\begin{aligned}(1) \quad & x^2 + y^2 = 4, \\(2) \quad & (x - 1)^2 + (y - 1)^2 = 1.\end{aligned}$$

We expand (2) and solve for x as follows.

$$\begin{aligned}x^2 - 2x + 1 + y^2 - 2y + 1 &= 1, \\x^2 + y^2 - 2x - 2y &= -1, \\4 - 2x - 2y &= -1, \text{ using (1),} \\x &= \frac{5}{2} - y.\end{aligned}$$

We now substitute $x = \frac{5}{2} - y$ in (1) and solve for y .

$$\begin{aligned}\left(\frac{5}{2} - y\right)^2 + y^2 &= 4, \\y^2 - 5y + \frac{25}{4} + y^2 &= 4, \\2y^2 - 5y + \frac{9}{4} &= 0.\end{aligned}$$

Finally, we apply the quadratic formula to get

$$y = \frac{5 \pm \sqrt{7}}{4}.$$

Therefore,

$$d = \frac{5 - \sqrt{7}}{4} \quad \& \quad b = \frac{5 + \sqrt{7}}{4}.$$

For a and c use $x = \frac{5}{2} - y$ to get

$$c = \frac{5 + \sqrt{7}}{4} \quad \& \quad a = \frac{5 - \sqrt{7}}{4}.$$

Notice that $(a, b) = (d, c)$. If you draw the line $x = y$ in the graph you'll see why this is. We only need b and d .

Here is the code for doing this integral in Maple.

```
> f := y -> 1-sqrt(2*y-y^2);  
> g := y -> sqrt(4-y^2);  
> h := y -> 1+sqrt(2*y-y^2);  
> b := (5+sqrt(7))/4;  
> d := (5-sqrt(7))/4;  
> int(int(2*x+y,x=f(y)..h(y)), y=0..d) +  
int(int(2*x+y,x=f(y)..g(y)), y=d..b);  
> evalf(%);
```

The result is 6.788647094.

Here is the code for doing this integral in Maxima.

```
(%i1) f:1-sqrt(2*y-y^2);  
(%i2) g:sqrt(4-y^2);  
(%i3) h:1+sqrt(2*y-y^2);  
(%i4) b:(5+sqrt(7))/4;  
(%i5) d:(5-sqrt(7))/4;  
(%i6) integrate(integrate(2*x+y,x,f,h),y,0,d) +  
integrate(integrate(2*x+y,x,f,g),y,d,b);  
(%i7) float(%);
```

The result is 6.788647093754035.