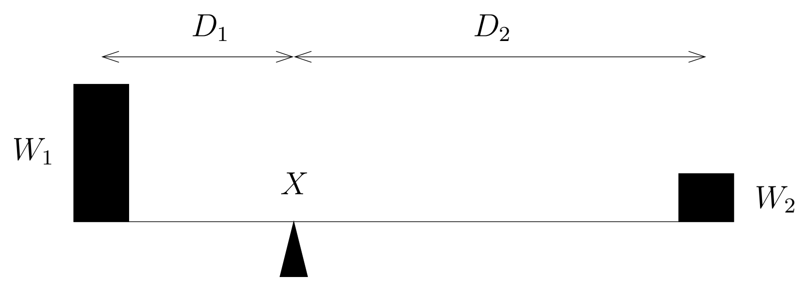
Archimedes' Law of the Lever

Theorem. Assume we have a strong thin rod with a weight at each end point. We wish to find the point in between where we can balance this system. Call the weights W_1 and W_2 and let D_1 and D_2 be their respective distances from some point X. Then the system will be balanced if and only if

$$W_1 D_1 = W_2 D_2.$$

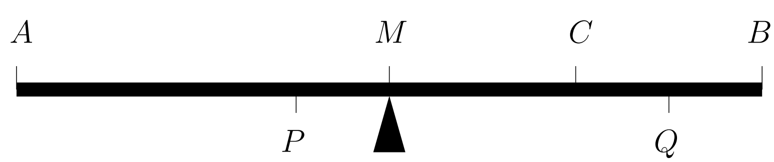
(We are assuming the rod is so thin we can ignore its weight.)



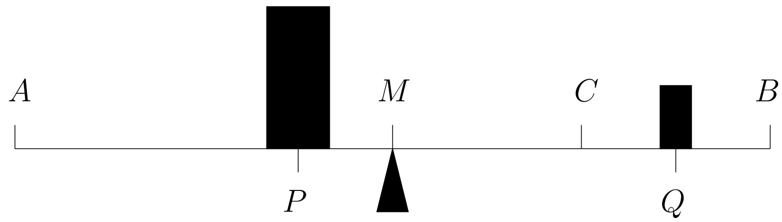
Proof. First we suppose we have a rod of uniform density d. Label the ends points A and B and the midpoint M. Then clearly M is the unique balance point.

NOTATION. For any two points, X and Y, on this rod let XY denote the length of the segment joining them and let $W(X, Y)$ be the weight of this segment.

Pick a point C somewhere between M and B. Let P be the midpoint of the segment from A to C and let Q be the midpoint of the segment from C to B.



Now imagine a new system with a very thin rod with a weight of $W(A, C)$ concentrated at P while a weight of $W(C, B)$ is concentrated at Q. The original point M is still the unique balance point.



The claim of the theorem is equivalent to saying

$$\frac{W(A,C)}{W(C,B)} = \frac{MQ}{PM}.$$

Think about this. We will derive this equation in three steps.

Step 1. Recall that d is the density of the original rod. Thus,

$$\frac{W(A,C)}{W(C,B)} = \frac{d \cdot AC}{d \cdot CB} = \frac{AC}{CB}.$$

Step 2. We compute as follows.

$$AC = AM + MC = \frac{AB}{2} + MC.$$

$$CB = AB - AC = AB - \left(\frac{AB}{2} + MC\right) = \frac{AB}{2} - MC.$$

Thus,

$$\frac{AC}{CB} = \frac{\frac{AB}{2} + MC}{\frac{AB}{2} - MC} = \frac{AB + 2MC}{AB - 2MC}.$$

Step 3. We compute as follows.

$$MQ = MC + CQ = MC + \frac{CB}{2} = MC + \frac{1}{2}\left(\frac{AB}{2} - MC\right) = \frac{MC}{2} + \frac{AB}{4}.$$

$$PM = AM - AP = \frac{AB}{2} - \frac{AC}{2} = \frac{AB}{2} - \frac{1}{2}\left(\frac{AB}{2} + MC\right) = \frac{AB}{4} - \frac{MC}{2}.$$

Thus,

$$\frac{MQ}{PM} = \frac{\frac{MC}{2} + \frac{AB}{4}}{\frac{AB}{4} - \frac{MC}{2}} = \frac{AB + 2MC}{AB - 2MC} = \frac{AC}{CB}.$$

We are done.