

Math 251
Practice Test 3
Most Answers Included

1. Let $\mathbf{F} = \langle x^3 + e^{-y} \sin z, x^2y + \arctan(z), \sqrt{y} \sec x \rangle$. Let V be the region of \mathbb{R}^3 bounded by the cylinder $z = 4 - x^2$, the plane $y + z = 5$, the xy -plane, and the xz -plane. Let S be the boundary of V . Find the flux of \mathbf{F} out through S .

Answer: 4608/35.

2. Find the work done by the force field $\mathbf{F} = \langle y, z, x \rangle$ in moving a particle from $(0, 0, 0)$ to $(1, 1, 1)$ along the path $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$.

Answer: 89/60.

3. In this problem we study linear vector fields. These are vector fields of the form

$$\mathbf{F} = \langle ax + by + cz, dx + ey + fz, gx + hy + kz \rangle.$$

(a) Under what conditions on the coefficients will \mathbf{F} be conservative?

(b) Suppose $\mathbf{F} = \langle 3x, -2y, z \rangle$. Write a potential function $f(x, y, z)$; *i.e.*, $\nabla f = \mathbf{F}$. What will type of surfaces will the level surfaces be?

4. Let $\mathbf{F} = \langle 0, 0, 5 \rangle$. Let S be the portion of the ellipsoid $4x^2 + 4y^2 + 100z^2 = 100$ above the xy -plane. Find the flux of \mathbf{F} up through S .

Hint: If \mathbf{F} is divergence free, $\nabla \cdot \mathbf{F} = 0$, you can replace S with a simpler surface.

Answer: 125π

5. Find the surface area obtained by rotating the function $y = \sin x$ for $x \in [0, \pi]$ about the x -axis.

Hint: See Formula 10 in 13.6. The integration is very hard.

The answer is ≈ 14.42 .

6. Find the center of mass of the hemispherical region below $z = \sqrt{25 - x^2 - y^2}$ and above the xy -plane with density function $\rho = z + 1$.

Answer: Mass is ≈ 752.673 . C.M. is $\approx (0, 0, 2.391)$.

7. Let $\mathbf{F} = \langle x^2z^3, 2xyz^3, xz^4 \rangle$. Let S be the boundary of the unit cube, $[0, 1] \times [0, 1] \times [0, 1]$, in \mathbb{R}^3 . Find the flux of \mathbf{F} out through S .

Answer: 1.

8. Let R be a bounded region of \mathbb{R}^3 that satisfies the conditions for the divergence theorem. Let $f(x, y, z)$ be a scalar function with all its second derivatives continuous. Let $\mathbf{F} = \nabla f$ and assume $\nabla \cdot \mathbf{F} = 0$. Prove that

$$\int \int_{\partial R} f \mathbf{F} \cdot \mathbf{n} \, dS = \int \int \int_R \mathbf{F} \cdot \mathbf{F} \, dV$$

where ∂R is the boundary of R .

9. Find the rotational inertia with respect to an axis of a solid sphere of radius R and uniform density k .

Answer: $\frac{8\pi k R^5}{15}$.

10. Let $\mathbf{F}(x, y) = \langle xy^3 + x, x^2y^2 \rangle$. (a) Find the work done in pushing an object counterclockwise once around the square in the xy -plane with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$. (b) Find the work done in pushing an object counterclockwise once around the triangle in the xy -plane with vertices $(0, 0)$, $(1, 1)$, $(0, 1)$.
Answers: (a) $-1/6$, (b) $-1/10$.

11. Compute $\oint_C (e^x \cos y) dx + (2x - e^x \sin y) dy$, where C is the boundary of the region in the xy -plane between the curves $y = x^2$ and $y = 8 - x^2$ going counterclockwise.
Answer: $128/3$.

12. Let S be a closed smooth surface enclosed a region R in \mathbb{R}^3 . Let $f(x, y, z)$ be a scalar function with continuous second partial derivatives. Prove that

$$\int \int \int_R \nabla^2 f dV = \int \int_S D_N f dS,$$

where $D_N f$ is the directional derivative of f in the direction of the outward unit normal vector to S . Hint: $\nabla^2 f = \nabla \cdot (\nabla f)$ and $D_N f = \nabla f \cdot N$.