Vector Calculus Differential Identities

Let f and g be scalar functions in three variables, (x, y, z). Let $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ and $\mathbf{G} = \langle G_1, G_2, G_3 \rangle$ be vector fields over $\mathbb{R}^3$. Let $\bigtriangledown = \langle \partial_x, \partial_y, \partial_z \rangle$. Then we make the following definitions.

- grad $f = \bigtriangledown f = \langle \partial_x f, \partial_y f, \partial_z f \rangle$.
- div $\mathbf{F} = \bigtriangledown \cdot \mathbf{F} = \partial_x F_1 + \partial_y F_2 + \partial_z F_3$.
- curl $\mathbf{F} = \bigtriangledown \times \mathbf{F} = \langle \partial_y F_3 - \partial_z F_2, \partial_z F_1 - \partial_x F_3, \partial_x F_2 - \partial_y F_1 \rangle$.
- $\bigtriangledown^2 f = \bigtriangledown \cdot (\bigtriangledown f) = \partial_{xx} f + \partial_{yy} f + \partial_{zz} f$. (It is called the *Laplacian*.)
- $\bigtriangledown^2 \mathbf{F} = \langle \bigtriangledown^2 F_1, \bigtriangledown^2 F_2, \bigtriangledown^2 F_3 \rangle$. (Vector field Laplacian.)
- $(\mathbf{F} \cdot \bigtriangledown) f = (F_1 \partial_x + F_2 \partial_y + F_3 \partial_z) f = F_1 \partial_x f + F_2 \partial_y f + F_3 \partial_z f$.
- $(\mathbf{F} \cdot \bigtriangledown) \mathbf{G} = \langle F_1 \partial_x G_1, F_2 \partial_y G_2, F_3 \partial_z G_3 \rangle$.

Then we have the following identities, provided the relavent derivatives exist and are continuous.

(1) $\bigtriangledown (f + g) = \bigtriangledown f + \bigtriangledown g$.

(2) $\bigtriangledown (fg) = (\bigtriangledown f) g + f (\bigtriangledown g)$.

(3) $\bigtriangledown \cdot (\mathbf{F} + \mathbf{G}) = \bigtriangledown \cdot \mathbf{F} + \bigtriangledown \cdot \mathbf{G}$.

(4) $\bigtriangledown \times (\mathbf{F} + \mathbf{G}) = \bigtriangledown \times \mathbf{F} + \bigtriangledown \times \mathbf{G}$.

(5) $\bigtriangledown (\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \bigtriangledown) \mathbf{G} + (\mathbf{G} \cdot \bigtriangledown) \mathbf{F} + \mathbf{F} \times (\bigtriangledown \times \mathbf{G}) + \mathbf{G} \times (\bigtriangledown \times \mathbf{F})$.

(6) $\bigtriangledown \cdot (f \mathbf{F}) = \bigtriangledown f \cdot \mathbf{F} + f (\bigtriangledown \cdot \mathbf{F})$.

(7) $\bigtriangledown \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\bigtriangledown \times \mathbf{F}) - \mathbf{F} \cdot (\bigtriangledown \times \mathbf{G})$.

(8) $\bigtriangledown \cdot (\bigtriangledown \times \mathbf{F}) = 0$.

(9) $\bigtriangledown \times (f \mathbf{F}) = \bigtriangledown f \times \mathbf{F} + f (\bigtriangledown \times \mathbf{F})$.

(10) $\bigtriangledown \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F} (\bigtriangledown \cdot \mathbf{G}) - \mathbf{G} (\bigtriangledown \cdot \mathbf{F}) + (\mathbf{G} \cdot \bigtriangledown) \mathbf{F} - (\mathbf{F} \cdot \bigtriangledown) \mathbf{G}$.

(11) $\bigtriangledown \times (\bigtriangledown f) = \mathbf{0}$.

(12) $\bigtriangledown \times (\bigtriangledown \times \mathbf{F}) = \bigtriangledown (\bigtriangledown \cdot \mathbf{F}) - \bigtriangledown^2 \mathbf{F}$.

(13) $\bigtriangledown \cdot (\bigtriangledown f \times \bigtriangledown g) = 0$.

Reference: Foundations of Electromagnetic Theory, by Reitz and Milford, 2^{nd} edition, Addison-Wesley, 1967.