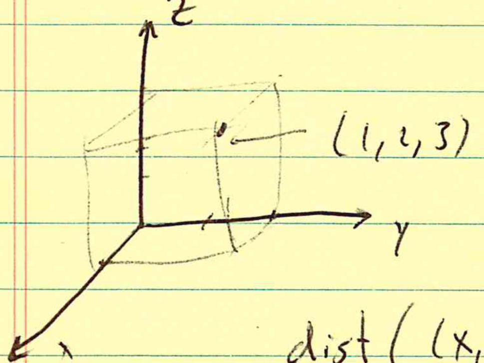
# 10.1 3-dimensional coordinate systems

$\mathbb{R}$ = real line, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x \in \mathbb{R},\ y \in \mathbb{R}\}$

$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) \mid x \in \mathbb{R},\ y \in \mathbb{R},\ z \in \mathbb{R}\}$




$$\text{dist}\big((x_1,y_1,z_1),(x_2,y_2,z_2)\big) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z}$$

Spheres: $x^2+y^2+z^2=R^2$ = points where dist to $(0,0,0)$ is $R$.

$(x-a)^2+(y-b)^2+(z-c)^2=R$

**Ex** Find the center and radius of $x^2+y^2+z^2=4x-2y$

**Sol**

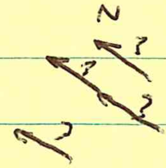
$$x^2-4x+4+y^2+2y+1+z^2=0+4+1$$

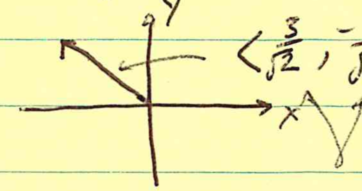
$$(x-2)^2+(y+1)^2+z^2=5$$

center: $(2,-1,0)$ Radius $=\sqrt{5}$.

## 10.2 Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

**Def** A vector is a direction and a magnitude.

**Ex** Go 3 miles NW. 


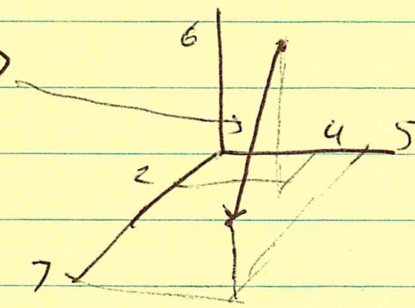


$\left\langle \frac{3}{\sqrt{2}}, \frac{-3}{\sqrt{2}} \right\rangle$

components

**Ex** The vector from $(2,4,6)$ to $(7,5,3)$ is

$$\langle 7-2, 5-4, 3-6 \rangle = \langle 5, 1, -3 \rangle$$



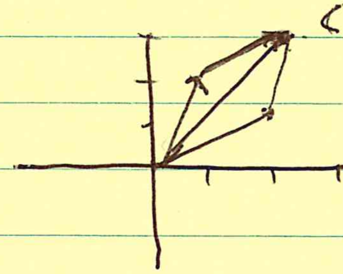

**Ex** What is the magnitude of $\langle 5,1,-3 \rangle$?

$$|\langle 5,1,-3 \rangle| = \sqrt{25+1+9} = \sqrt{35}$$

### Vector Addition

$$\langle a,b \rangle + \langle c,d \rangle = \langle a+c, b+d \rangle$$
$$\langle a,b,c \rangle + \langle d,e,f \rangle = \langle a+d, b+e, c+f \rangle.$$

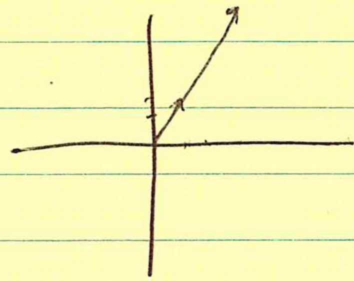
**Ex** $\langle 1,2 \rangle + \langle 2,1 \rangle = \langle 3,3 \rangle$

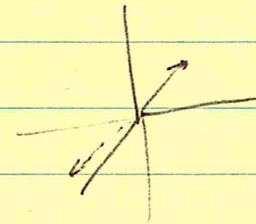


# Scalar Multiplication

$$r\langle a,b\rangle = \langle ra, rb\rangle \qquad r\langle a,b,c\rangle = \langle ra, rb, rc\rangle.$$

**Ex** $\quad 5\langle 1,2\rangle = \langle 5,10\rangle \qquad -\langle 0,1,1\rangle = \langle 0,-1,-1\rangle$





**Fact** $\quad |r\,v| = |r|\,|v|.$ pf: $|r\langle a,b,c\rangle| = |\langle ra, rb, rc\rangle|$

$$= \sqrt{r^2a^2 + r^2b^2 + r^2c^2} = \sqrt{(r^2)(a^2+b^2+c^2)}$$

$$= |r|\sqrt{a^2+b^2+c^2}$$

$$= |r|\,|\langle a,b,c\rangle|.$$

**Notation** $\quad \mathbf{0} = \langle 0,0\rangle$ or $\langle 0,0,0\rangle$.

<u>Properties</u> Let $u$, $v$, and $w$ be vectors. Let $s, t$ be real numbers.

1. $u+v = v+u$
2. $u+(v+w) = (u+v)+w$
3. $u+\mathbf{0} = u$
4. $u+(-u) = \mathbf{0}$
5. $s(u+v) = su+sv$
6. ~~(s(tu)) = (st)~~ $(s+t)u = su+tu$
7. $s(tu) = (st)u$
8. $1 \cdot u = u$, $0u = \mathbf{0}$

<u>Pf of 2 for $\mathbb{R}^2$.</u> Let $u = \langle u_1, u_2 \rangle$, $v = \langle v_1, v_2 \rangle$ $w = \langle w_1, w_2 \rangle$

$$u+(v+w) = \langle u_1, u_2 \rangle + (\langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle)$$

$$= \langle u_1, u_2 \rangle + \langle v_1 + w_1, v_2 + w_2 \rangle$$

$$= \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2) \rangle$$

$$= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2 \rangle$$

$$= \langle u_1 + v_1, u_2 + v_2 \rangle + \langle w_1, w_2 \rangle$$

$$= (\langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle) + \langle w_1, w_2 \rangle$$

$$= (u+v) + w$$

Notation $i = \langle 1, 0, 0 \rangle$, $j = \langle 0, 1, 0 \rangle$, $k = \langle 0, 0, 1 \rangle$

Ex $\langle 5, 2, 7 \rangle = 5i + 2j + 7k$.

Def $v$ is a unit vector if $|v| = 1$.

Ex Find the unit vector $u$ with the same direction as $v = \langle 1, 2, 1 \rangle$.

Sol $|\langle 1, 2, 1 \rangle| = \sqrt{1+4+1} = \sqrt{6}$.

$$u = \frac{v}{|v|} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

## 10.3 The dot product

**Def** Let $v = \langle v_1, v_2, v_3 \rangle$ , $w = \langle w_1, w_2, w_3 \rangle$.

Then $v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3 \in \mathbb{R}$.

**Fact** $|v| = \sqrt{v \cdot v}$

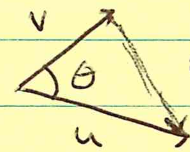
**Ex** $(2i + 2j - k) \cdot (3i + j + k) = 6 + 2 - 1 = 7.$

$\langle 3, 4, 2 \rangle \cdot \langle 1, 1, 2 \rangle = 3 + 4 + 4 = 11$

Properties Let $u$, $v$ and $w$ be vectors and $r \in \mathbb{R}$.

1. $v \cdot v = |v|^2$
2. $v \cdot w = w \cdot v$
3. $u \cdot (v + w) = u \cdot v + u \cdot w$
4. $(ru) \cdot v = r(u \cdot v) = u \cdot (rv)$
5. $\mathbf{0} \cdot v = 0$ the number.

# Angles




$w = u - v$

$v + w = v + u - v = u$
$\Rightarrow w = u - v.$

**Thm** $v \cdot u = |v||u| \cos\theta$

**Pf** Recall the Law of Cosines:

$$|u-v|^2 = |v|^2 + |u|^2 - 2|v||u| \cos\theta$$ (This is P. thm when $\theta = \frac{\pi}{2}$.

But we also have from the properties of vectors that

$$|u-v|^2 = (u-v)\cdot(u-v) = (u-v)\cdot u + (u-v)\cdot(-v)$$

$$= u\cdot u - v\cdot u - u\cdot v + v \cdot v$$

$$= |u|^2 - 2(v\cdot u) + |v|^2.$$

Thus

$$|u|^2 - 2(v\cdot u) + |v|^2 = |v|^2 + |u|^2 - 2|v||u|\cos\theta$$

implies $$v\cdot u = |v||u|\cos\theta. \qquad \square$$

This is also written as $\cos\theta = \dfrac{v\cdot u}{|v||u|}$

**Ex** Find the angle between $\langle 1,2,1\rangle$ and $\langle 0,1,1\rangle$.

**Sol**
$$\cos\theta = \frac{1\cdot 0 + 2\cdot 1 + 1\cdot 1}{\sqrt{1+4+1}\,\sqrt{0+1+1}} = \frac{3}{\sqrt{6}\,\sqrt{2}} = \frac{\sqrt{3}}{2}$$

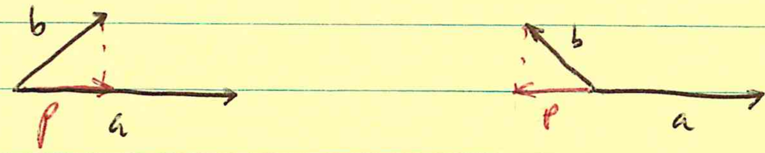
Thus, $\theta = \frac{\pi}{6}$ or $30°$.

**Note** $v \perp u$ means $v$ is perpendicular to $u$.
Then $\theta = \frac{\pi}{2}$ so $v \cdot u = 0$.

We define $v \perp 0$ for any $v$. Then

$$v \perp u \iff v \cdot u = 0$$

## Projections

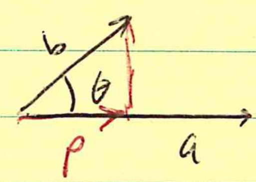



$\text{proj}_a b$ = projection of b onto a.

$\text{comp}_a b$ = mag. and sign of $\text{proj}_a b$.

**Thm** (i) $\text{proj}_a b = \dfrac{a \cdot b}{|a|^2} a$. (ii) $\text{comp}_a b = \dfrac{a \cdot b}{|a|}$.

**pf** (ii) follows from (i).

(i) 


$\pm |p| = |b| \cos\theta$

$$p = |b| \cos\theta \frac{a}{|a|} = |b| \frac{a \cdot b}{|a||b|} \frac{a}{|a|}$$

$$= \frac{a \cdot b}{|a|^2} a$$

$$= \underbrace{\left(\frac{a \cdot b}{|a|}\right)}_{\text{comp}_a b} \frac{a}{|a|}$$

**Ex** project $\langle 1, 2, 1 \rangle$ onto $\langle 1, 1, 1 \rangle$.

**Sol** ~~a·b = 4~~ $\langle 1, 2, 1 \rangle \cdot \langle 1, 1, 1 \rangle = 4$

~~$|a| = \sqrt{6}$ $|b| = \sqrt{3}$ proj~~ a $|\langle 1, 1, 1 \rangle| = \sqrt{3}$

$$p = \frac{4}{3} \langle 1, 1, 1 \rangle = \left\langle \frac{4}{3}, \frac{4}{3}, \frac{4}{3} \right\rangle.$$