11.2 Limits and Continuity

Def $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ means (roughly) that as (x,y) gets near to (a,b), the value of $f(x,y)$ approaches L. More precisely, $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$$\text{if } 0 < \text{dist}((x,y),(a,b)) < \delta$$
$$\text{then } |f(x,y) - L| < \varepsilon.$$

Def $f(x,y)$ is continuous at (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists and equals $f(a,b)$.

Facts If $g(x)$ and $h(y)$ are continuous one variable functions then $f(x,y) = g(x)\,h(y)$ is continuous. Sums, products, and compositions of cont. mult. variable functions are cont. So are quotients as long as the denominator is not zero.

Ex $x^3y + 3x^2y^2 + 7x$, $x^4 y \sin(x+y^2)$, $\dfrac{x^2y+7y^3}{x^2+y^2+1}$, $y^3 x e^{7xy}$ are continuous.

Many concepts from limits of single variable functions extend to mult. variable functions.

Recall If $f(x)$ is undefined at $x=c$, but $f(x)=h(x)$ near c and $h(x)$ is cont. at $x=c$, then $\lim_{x\to c} f(x)=h(c)$.
For example

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2+2x-3} = \lim_{x\to 1} \frac{(x+2)(x-1)}{(x+3)(x-1)} = \lim_{x\to 1} \frac{x+2}{x+3} = \frac{3}{4}.$$

This idea can be applied to functions of two or more variables.

Ex Let $f(x,y) = \dfrac{x^3+xy^2+x^2+y^2}{x^2y+y^3-2x^2-2y^2}$. $f(0,0) = \text{“}\frac{0}{0}\text{”}$ is undefined

Yet,

$$\operatorname{limit}_{(x,y)\to(0,0)} f(x,y) = \operatorname{limit} \frac{(x^2+y^2)(x+1)}{(x^2+y^2)(y-2)} = \lim_{(x,y)\to(0,0)} \frac{x+1}{y-2} = -\frac{1}{2}$$

Recall If $\operatorname{limit}_{x\to c} |f(x)| = 0$ then $\operatorname{limit}_{x\to c} f(x)=0$. This carries over the multivariable functions. Also, the Squeeze Thm. If $f(x) \le g(x) \le h(x)$ and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\operatorname{limit}_{x\to c} g(x) = L$.

In the next example we apply these two ideas to a function of two variables.

Ex Let $f(x,y) = \dfrac{x^2 y}{x^2+y^2}$. Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

Sol. Now $f(0,0) =$ "$\frac{0}{0}$" is not defined. But notice

$$0 \leq \frac{x^2}{x^2+y^2} \leq 1.$$

Thus $$0 \leq \left|\frac{x^2 y}{x^2+y^2}\right| = \frac{x^2}{x^2+y^2}|y| \leq |y|.$$

As $(x,y)\to(0,0)$, clearly $|y|\to 0$. Therefore,

$$\lim_{(x,y)\to(0,0)} \left|\frac{x^2 y}{x^2+y^2}\right| = 0 \quad \text{by the Squeeze Thm}$$

and thus,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0.$$

Application Define $g(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is not } (0,0) \\ 0 & \text{if } x=y=0. \end{cases}$

Then, $g(x,y)$ is continuous! We removed the singularity where $f(x,y)$ was undefined, and created a cont. func.

Recall

If $\lim_{x \to c} f(x)$ exists, then $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$,

that is the limits from the left and right match.

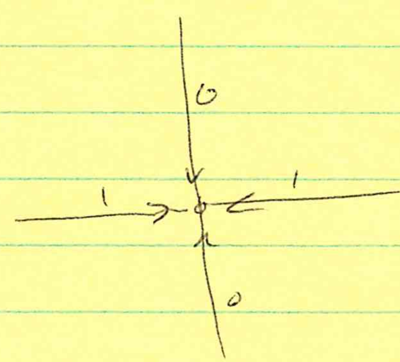
With multivariable functions there are many

more paths that (x,y) could travel en route

to (a,b). All must match for the

limit to exist.

Ex Let $f(x,y) = \frac{x^2}{x^2+y^2}$. Hold $x=0$, and take the limit as $y \to 0^\pm$. Since $f(0,y)=0$, we ~~can~~ have $\lim_{y\to 0^\pm} f(0,y) = \lim_{y \to 0^\pm} 0 = 0$. But now hold $y=0$ and take the limit as $x \to 0^\pm$. $f(x,0) = \frac{x^2}{x^2} = 1$. Thus $\lim_{x \to 0^\pm} f(x,0) = 1$. Thus while limits ~~from~~ along the pos. and neg x-axis match, and the pos & neg y-axis match, they do not match each other, so $\lim_{(x,y)\to(0,0)} f(x,y)$ d.n.e.



This function has a "tear" at the origin. We cannot assign it a value at $(0,0)$ that would make it continuous.

There exist functions where every linear path gives the same ~~value for~~ limiting value, but there are curved paths that don't match. ~~these~~.

Ex Let $f(x,y) = \dfrac{x^3 y}{x + y^3}$.

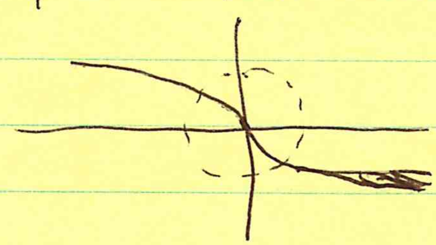
Clearly,

$$\lim_{y \to 0} f(0,y) = 0 \qquad \lim_{x \to 0} f(x,0) = 0.$$

In fact if we let $y = mx$ and take $x \to 0$, we get

$$f(x, mx) = \frac{m x^4}{x + m^3 x^3} = \frac{m x^3}{1 + m^3 x^2} \to 0 \text{ as } x \to 0.$$

Yet $f(x,y)$ is not even defined along the curve $x = -y^3$. So, the limit d.n.e.



These ideas carry ~~alt~~ over to functions of three variables

Ex $f(x,y,z) = \dfrac{xy+yz+xz}{3x^2+2y^2+z^2}$. Study limit as $(x,y,z) \to ($

Along any axis, the limit is zero. For example let $x=y=0$ and take the limit as $z \to 0$.

$$\lim_{z \to 0} f(0,0,z) = \lim \frac{0}{z^2} = 0$$

But take the limit along the line $x=y$, $z=0$.

$$\lim_{x \to 0} f(x,x,0) = \lim \frac{x^2}{3x^2} = \frac{1}{3}$$

Take the limit along the line $x=y=z$.

$$\lim_{x \to 0} \frac{3x^2}{6x^2} = \frac{1}{2}.$$

Thus the general limit d.n.e.

Ex Use polar coordinates to show that

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$$

Sol $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$.

Thus $$\frac{3x^2y}{x^2+y^2} = \frac{3r^3\cos^2\theta\sin\theta}{r^2} = 3r\cos^2\theta\sin\theta$$

Since $-1 \le \cos^2\theta\sin\theta \le 1$ we know

$$-3r \le 3r\cos^2\theta\sin\theta \le 3r.$$

As $(x,y)\to(0,0)$, $r\to 0$. Thus, by the ~~sq~~ Squeeze Theorem

$$\lim_{r\to 0} 3r\cos^2\theta\sin\theta = 0$$

This is the same as

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$$

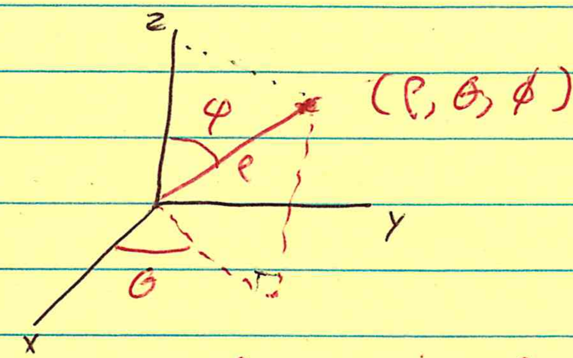
[Optional Reading]

Ex Use spherical coordinate to show that

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = 0.$$

Sol We have not covered spherical coordinates yet so this is <u>optional reading</u>.

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$
$$\rho = x^2+y^2+z^2$$



(ρ = rho, the Greek letter)

Now

$$\frac{xyz}{x^2+y^2+z^2} = \frac{\rho^3 \sin^2\phi \cos\theta \sin\theta \cos\phi}{\rho^2}$$

$$= \rho \sin^2\phi \cos\theta \sin\theta \cos\phi.$$

Since $-1 \le \sin^2\phi \cos\theta \sin\theta \cos\phi \le 1$ we know

$$-\rho \le \rho \sin^2\phi \cos\theta \sin\theta \cos\phi \le \rho.$$

Thus the limit as $\rho \to 0$ of the middle term is 0 by The Squeeze Thm. Hence,

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = 0.$$