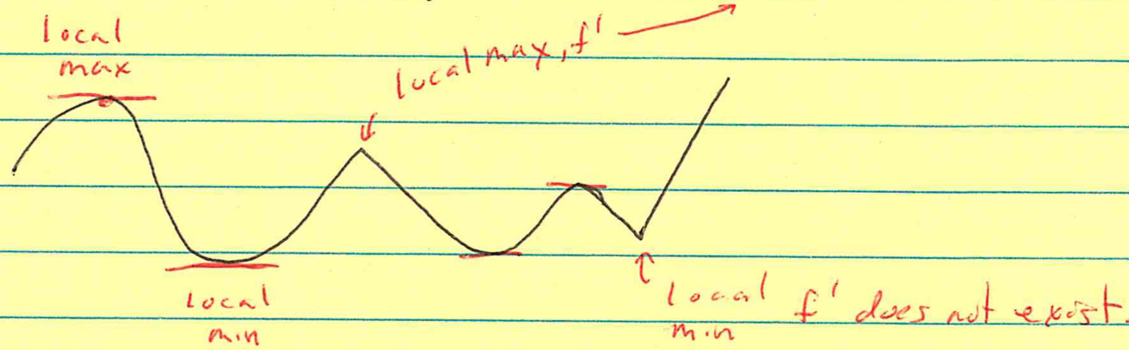
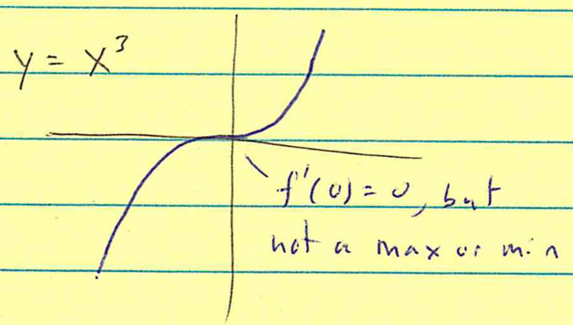
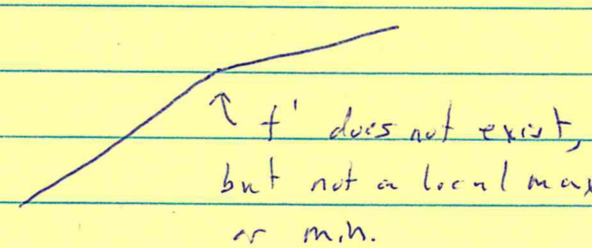
# 11.7 Maximums and Minimums

**Recall** The First Derivative Test: If $y = f(x)$ has a local (or relative) max or min at $x = x_0$ then either $f'(x_0) = 0$ or it does not exist.

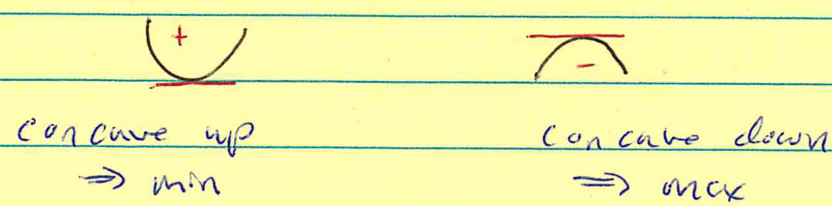



But just because $f'(x_0) = 0$ or does not exist, does not mean $f$ has to have a min or max at $x_0$




**Recall** The Second Der. Test: If $x_0$ is a local exterma (min or max) at $f(x)$ and $f'(x_0) = 0$, then if $f''(x_0) > 0$ it is a min and if $f''(x_0) < 0$ it is a max.



**Def** Let $f(x,y)$ be a function of two variables.

- ~~Then~~ $f$ has an <u>absolute</u> (or <u>global</u>) <u>max</u> at $(a,b)$ if $f(x,y) \le f(a,b)$ for all $(x,y)$ in the domain of $f$.
- $f$ has a <u>relative</u> (or <u>local</u>) <u>max</u> at $(a,b)$ if $f(x,y) \le f(a,b)$ for all $(x,y)$ in an open disk centered at $(a,b)$.
- The definitions of abs. and rel. minimums are similarly

**Thm** If $f(x,y)$ has a local max (or min) at $(a,b)$ then $\nabla f(a,b) = \langle 0,0 \rangle$ or does not exist.

**Outline of Pf** Let $g(x) = f(x,b)$. Then $g(x)$ has a local max (or min) at $x=a$. ~~Then~~ Thus,

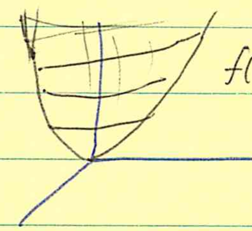
$$\frac{dg}{dx}(a) = 0 \text{ or does not exist.}$$

But $\frac{dg}{dx} = \frac{\partial f}{\partial x}$. Thus $\frac{\partial f}{\partial x} = 0$ or does not exist.
Similarly we can show $\frac{\partial f}{\partial y} = 0$ or does not exist. □

**Def** The points where $\nabla f = \langle 0,0 \rangle$ or d.n.e. are called the <u>critical points</u> of $f$.

Ex  Let $f(x,y) = x^2+y^2$. Find the local extrema.

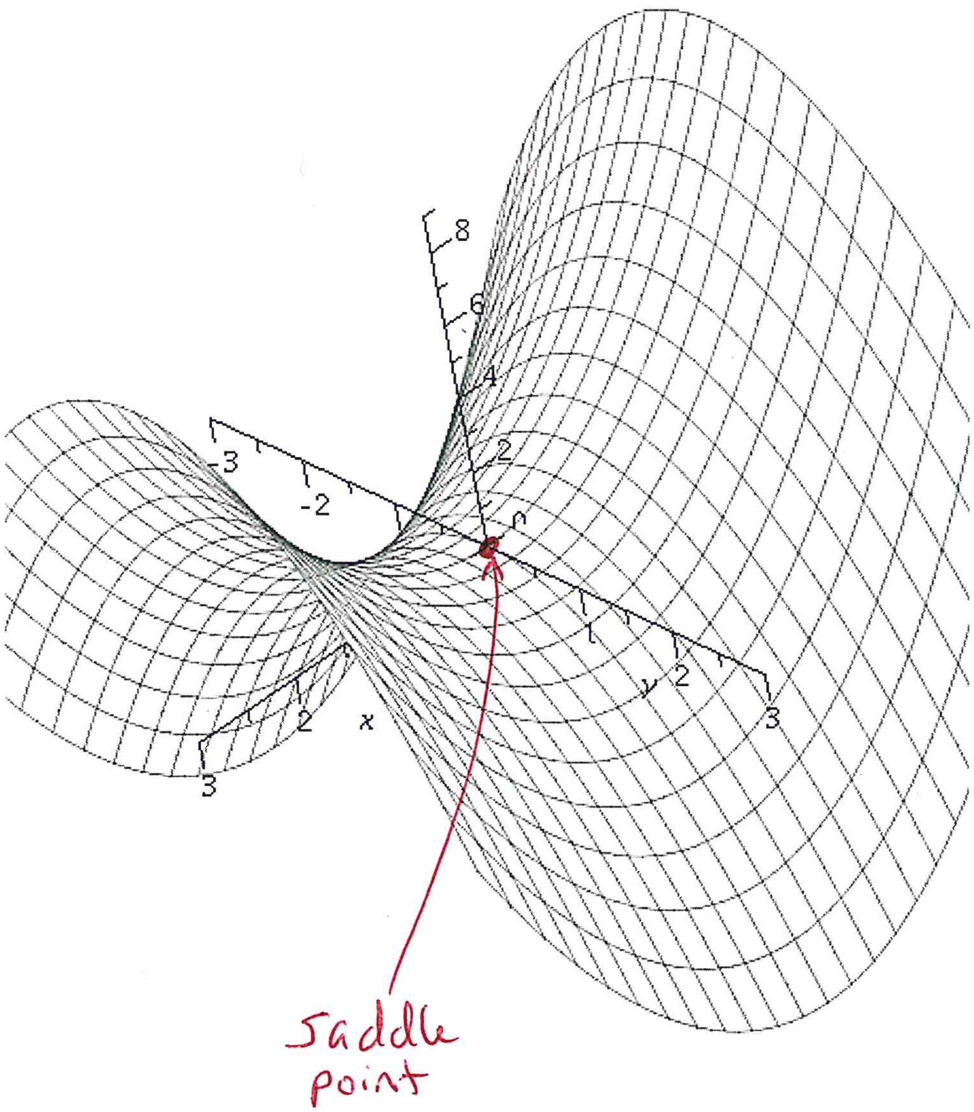
Sol  $\nabla f = \langle 2x, 2y \rangle$. Thus $\nabla f = \langle 0,0 \rangle$ only at $(0,0)$ and it always exists. You can check graphically that it is a local, and in fact the global, minimum.



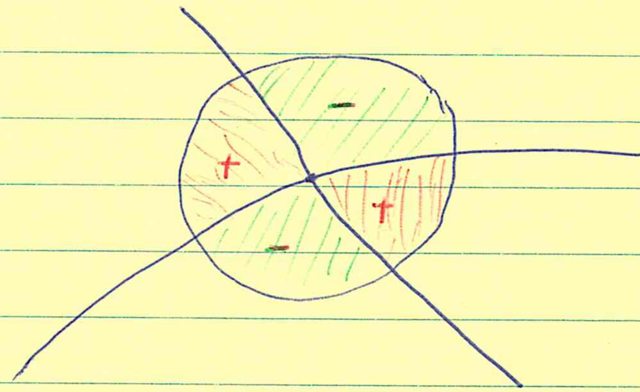

Ex  Let $f(x,y) = x^2 - y^2$. Find the critical points. Are they mins or maxs?

Sol  $\nabla f = \langle 2x, -2y \rangle = \langle 0,0 \rangle$ only at $(0,0)$ and is always defined. But this function does not have a min or a max at $(0,0)$ or any where else. The point $(0,0)$ is called a saddle point of $f(x,y)$.

$f(x,y) = x^2 - y^2$



**Def** $f(x,y)$ has a saddle point at $(a,b)$ if there exists a pair of curves crossing at $(a,b)$ such that in some disk centered at $(a,b)$ we have $f(x,y) \geq f(a,b)$ in two opposite sectors and $f(x,y) \leq f(a,b)$ in the other two sectors.




(I'm using + to mean above $f(a,b)$ and − to mean below $f(a,b)$.)

This is a new phenomenon that does not occur in graphs of single variable functions.

**Recall** $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

**Thm** The Second Derivative Test [Know this!]

Let $f(x,y)$ have continuous second partial derivatives in an open disk containing $(a,b)$ at which $\nabla f = (0,0)$. ~~Let~~ Let

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 \quad \text{at } (a,b).$$

1. If $D > 0$ and $f_{xx} > 0$, then $f$ has a local min at $(a,b)$
2. If $D > 0$ and $f_{xx} < 0$, then $f$ has a local max at $(a,b)$
3. If $D < 0$, then $f$ has a saddle point at $(a,b)$.
4. If $D = 0$, then the test is inconclusive.

We will not prove this, but by examining some key examples you can get a rough feel for why it makes sense.

**Ex** Let $f(x,y) = x^4 + y^4$. Then $\nabla f = \langle 4x^3, 4y^3 \rangle$ and $(0,0)$ is the only critical point. But $f_{xx} = 12x^2$, $f_{yy} = 12x^2$ and $f_{xy} = 0$ are all $0$ at $(0,0)$. Thus $D = 0$. But we can see that $(0,0)$ gives a local (in fact global) min. [Compare this with what happens with the one variable 2nd der. test for $y = x^4$.]

Notice $D > 0$ and $f_{xx} > 0 \Rightarrow f_{yy} > 0$.
$D > 0$ and $f_{xx} < 0 \Rightarrow f_{yy} < 0$.
So $D > 0 \Rightarrow f_{xx}$ and $f_{yy}$ have the same sign.

If $D < 0$ either $f_{xx}$ and $f_{yy}$ have opposite signs or $(f_{xy})^2$ is large.

<u>Ex</u> Let $f(x,y) = ax^2 + bxy + cy^2$.

Then $\nabla f = \langle 2ax + by, bx + 2cy \rangle = \langle 0,0 \rangle$ at $(0,0)$.

Now $f_{xx} = 2a$, $f_{yy} = 2b$ and $f_{xy} = b$.

Thus $D = 4ac - b^2$

If $a > 0$, $c > 0$ and $b$ is "small" we have $D > 0$ and $(0,0)$ is a local min.

If $a < 0$, $c < 0$ and $b$ is "small" we have $D > 0$ and $(0,0)$ is a local max.

If $a = c = 0$ and $b \neq 0$, then $D < 0$ and we have a saddle at $(0,0)$.

Study the graphs of $ax^2 + bxy + cy^2$ for various values of $a$, $b$ and $c$ and you'll see why the 2nd der. test makes sense.

Ex A "bad case." Let $f(x,y) = x^2+2xy+y^2$.
Then $D = 4\cdot1\cdot1-2^2=0$, everywhere.

Notice $f(x,y) = (x+y)^2$

So, $f(x,y) \geq 0$ for all $x,y$, and $f(0,0)=0$.
So, $(0,0)$ is a local min. But $f(x,y)=0$ whenever $y=-x$. So, there ~~is~~ are infinitely many points where $f(x,y)$ has a local min.

Optional Note One way to prove the Second Der. Test is valid is to study the 2 variable Taylor poly of $f(x,y)$ centered at a critical pt $(a,b)$. The ~~quadra~~ quadratic terms, ~~$x^2, xy, y^2$ give~~ give a quadratic surface that approximates the given surface $z=f(x,y)$ near $(a,b)$.

~~An Example from class for 11.7.~~

**Ex 1** Let $f(x,y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$.
Find the mins, maxs and saddle points of the surface $z = f(x,y)$.

Step 1: Find the critical points.

$$f_x = -6yx - 6x = -6x(y+1)$$

$$f_y = 3y^2 - 3x^2 - 6y$$

$$f_x = 0 \iff x(y+1) = 0 \iff x = 0 \text{ or } y = -1.$$

Suppose $x=0$. Then $f_y = 3y^2 - 6y = 3y(y-2)$.
Thus $f_y = 0$ when $y=0$ or $y=2$. Thus

$$(0,0) \text{ and } (0,2)$$

are critical points.

Suppose $y=-1$. Then $f_y = 3 - 3x^2 + 6 = 3(3 - x^2)$.
Hence $f_y = 0$ when $x = \pm\sqrt{3}$. Thus

$$(\sqrt{3}, -1) \text{ and } (-\sqrt{3}, -1)$$

are also critical points.

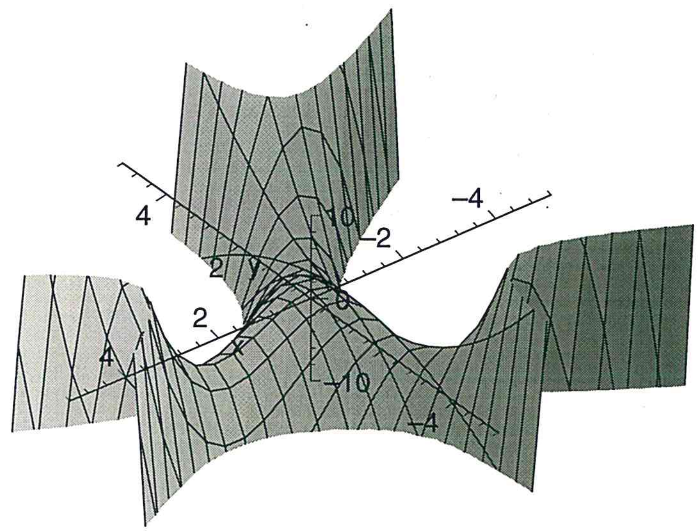
## Step 2: Apply 2nd-Derivative Test.

$$f_{xx} = -6y - 6 = -6(y+1)$$
$$f_{yy} = 6y - 6 = 6(y-1)$$
$$f_{xy} = f_{yx} = -6x$$

Next we compute those and $D = f_{xx}f_{yy} - (f_{xy})^2$ at each critical point. I like to use a table.

| | $(0,0)$ | $(0,2)$ | $(\sqrt{3},-1)$ | $(-\sqrt{3},-1)$ |
|---|---|---|---|---|
| $f_{xx}$ | $-6$ | $-18$ | $0$ | $0$ |
| $f_{yy}$ | $-6$ | $6$ | $-12$ | $-12$ |
| $f_{xy}$ | $0$ | $0$ | $-6\sqrt{3}$ | $6\sqrt{3}$ |
| $D$ | $36$ | $-108$ | $-108$ | $-108$ |
| conclusion | Max | Saddle | Saddle | Saddle |

```
> plot3d(y^3-3*y*x^2-3*y^2-3*x^2+1,x=-5..5,y=-5..5,view=-10..10);
```

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>
```

**Ex2** Let $f(x,y) = x^3 + x^2y - x^2 - 3y^2 + x + 2y$.

Find and classify the extrema of $f$.

$$f_x = 3x^2 + 2xy - 2x + 1 = 3x^2 + 2(y-1)x + 1$$

$$f_y = x^2 - 6y + 2$$

$$f_y = 0 \Rightarrow y = \frac{x^2+2}{6} = \frac{1}{6}x^2 + \frac{1}{3}$$

substitute into $f_x = 0$ to get

$$3x^2 + 2\left(\frac{1}{6}x^2 - \frac{2}{3}\right)x + 1 = 0$$

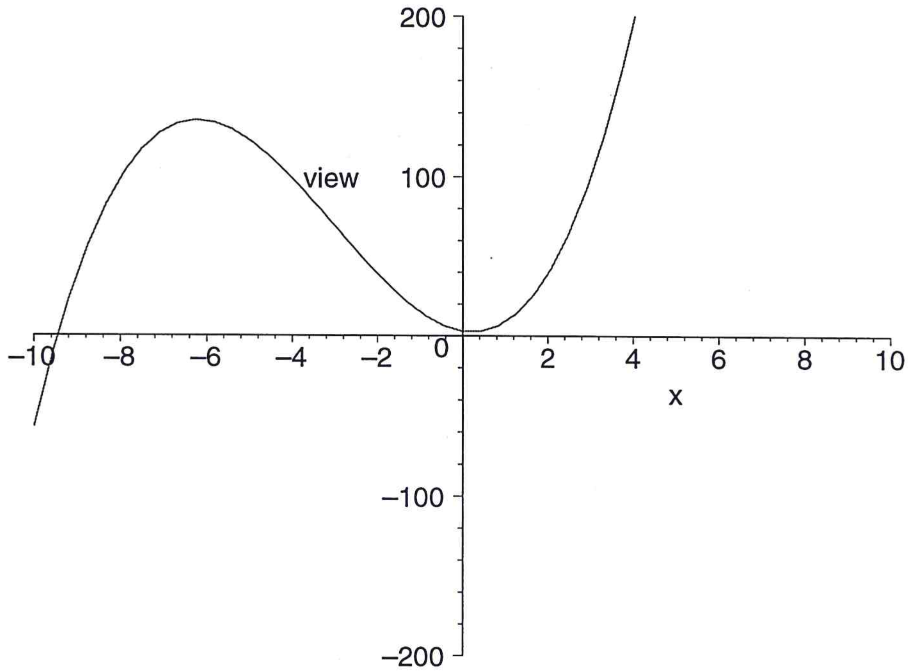
$$\frac{1}{3}x^3 + 3x^2 - \frac{4}{3}x + 1 = 0$$

$$x^3 + 9x^2 - 4x + 3 = 0$$

I did not see how to factor this. I graphed it with Maple and could see that there is just one real zero at about $-9.5$.

I used the command fsolve to get a better estimate of the zero. See next page.

```
> plot(x^3+9*x^2-4*x+3,x=-10..10,view=-200..200);
```




```
> x:=fsolve(x^3+9*x^2-4*x+3=0);
```

$x := -9.456535174$

```
> y:=((-9.456535174)^2+2)/6;
```

$y := 15.23767625$

```
> fxx:=6*x+2*y-2;
```

$fxx := -28.26385854$

```
>
```

$x = -9.456535174$ $y = 15.2376725$

is only c.p.

$f_{xx} = 6x + 2y - 2 = -28.26385854$

$f_{yy} = -6$

$f_{xy} = f_{yx} = 2x = -18.913070348$

$D = (f_{xx})(f_{yy}) - (f_{xx})^2 = -188.14525 < 0.$

It is a saddle

**Ex** Let $f(x,y) = x^2 - y^2 - 2xy - 4x$.
Find and classify the critical points of $f(x,y)$.

**Sol** Step 1: Find the critical points.

$f_x = 2x - 2y - 4$. $f_y = -2y - 2x$.

We need to solve the system of linear equations

$$\left.\begin{aligned} 2x - 2y - 4 &= 0 \\ -2y - 2x &= 0 \end{aligned}\right\} \iff \left.\begin{aligned} x - y &= 2 \\ x + y &= 0 \end{aligned}\right\} \Rightarrow 2x = 2 \Rightarrow x = 1.$$

Since $x = 1$ and $x + y = 0$, we get that $y = -1$. Thus, there is just one critical point, $(1,-1)$.

Step 2: Apply Second Der. Test.

$f_{xx} = 2$, $f_{yy} = -2$, $f_{xy} = -2 \Rightarrow D = -8$.

This is for all $(x,y)$, so clearly $D(1,-1) = -8 < 0$ and we conclude that $(1,-1)$ is a saddle point.

**Ex** Let $f(x,y) = x^3 + y^3 + 3xy$.
Find and classify the critical points of $f(x,y)$.

**Sol** Step 1: Find critical points.

$f_x = 3x^2 - 3y$, $f_y = 3y^2 - 3x$.

$$\left.\begin{aligned} f_x = 0 &\Leftrightarrow x^2 = y \\ f_y = 0 &\Leftrightarrow y^2 = x \end{aligned}\right\} \Rightarrow x^4 = x \Rightarrow x = 0 \text{ or } 1.$$

If $x = 0$, then $y = 0^2 = 0$. If $x = 1$, then $y = 1^2 = 1$.
Thus, $(0,0)$ and $(1,1)$ are the two critical points.

Step 2: Apply Second Derivative Test.

$f_{xx} = 6x$ $f_{yy} = 6y$, $f_{xy} = -3$.
Thus, $D(x,y) = 36xy - 9$.

$D(0,0) = -9 < 0$. Thus $(0,0)$ is a saddle point.

$D(1,1) = 36 - 9 = 25 > 0$. We check $f_{xx}(1,1) = 6 \cdot 1 = 6 > 0$.
Thus, $(1,1)$ is a local minimum.

**Ex** Let $f(x,y) = e^x + \ln y^4 - x - y$. Assume $y \neq 0$. Find and classify the critical points.

**Sol** Step 1: Find the critical points

$f_x = e^x - 1 \qquad f_y = \frac{1}{y^4}(4y^3) - 1 = \frac{4}{y} - 1.$

$f_x = 0 \Rightarrow e^x = 1 \Rightarrow x = 0.$
$f_y = 0 \Rightarrow \frac{4}{y} = 1 \Rightarrow y = 4.$

Thus $(0, 4)$ is the only critical point.

Step 2: Apply the Second Derivative Test

$f_{xx} = e^x \qquad f_{yy} = \frac{-4}{y^2} \qquad f_{xy} = 0.$
$f_{xx}(0,4) = e^0 = 1$
$f_{yy}(0,4) = -4/4^2 = -\frac{1}{4}.$

Thus $D(0,4) = 1(-\frac{1}{4}) - 0^2 = -\frac{1}{4} < 0.$

Hence $(0, 4)$ is a saddle point.

**Ex** Let $f(x,y) = \sin(\pi x)\sin(\pi y)$. Find the critical point and determine which are mins, maxs and saddles.

**Sol** Step 1: Find critical points.

$f_x = \pi\cos(\pi x)\sin(\pi y) = 0$ if $y$ is an integer or $x$ is an odd multiple of $\frac{1}{2}$

$f_y = \pi\sin(\pi x)\cos(\pi y) = 0$ if $x$ is an integer or $y$ is a odd multiple of $\frac{1}{2}$.

(Graph the sin and cos functions so that you see this.)

Thus, the critical points occur when $x$ and $y$ are both integers or when $x$ and $y$ are both odd multiples of $\frac{1}{2}$. (Don't just take my word for it, think it through

Step 2: Apply Second Derivative Test

$f_{xx} = -\pi^2\sin(\pi x)\sin(\pi y)$.

$f_{yy} = -\pi^2\sin(\pi x)\sin(\pi y)$.

$f_{xy} = \pi^2\cos(\pi x)\cos(\pi y)$.

$D = \pi^4\left(\sin^2(\pi x)\sin^2(\pi y) - \cos^2(\pi x)\cos^2(\pi y)\right)$.

If $x$ and $y$ are both integers $D = \pi^4(0^2 0^2 - (\pm 1)^2(\pm 1)^2)$
$= -\pi^4$.

Thus, these are saddle points.

If $x$ and $y$ are odd multiples of $\frac{1}{2}$, then

$$D = \pi^4\left( (\pm 1)^2(\pm 1)^2 - 0^2 0^2 \right) = \pi^4 > 0.$$

We need to check $f_{xx} = -\pi^2 \sin(\pi x)\sin(\pi y)$ for

$x = \frac{2m+1}{2}$ and $y = \frac{2n+1}{2}$, where $m$ and $n$ are integers.

We look at examples and try to find a pattern.

$f_{xx}\left(\frac{1}{2}, \frac{1}{2}\right) = -\pi^2 < 0$ max
$f_{xx}\left(\frac{1}{2}, \frac{3}{2}\right) = \pi^2 > 0$ min
$f_{xx}\left(\frac{1}{2}, \frac{5}{2}\right) = -\pi^2 < 0$ max
$f_{xx}\left(\frac{1}{2}, \frac{7}{2}\right) = \pi^2 > 0$ min
etc

$f\left(\frac{3}{2}, \frac{1}{2}\right) = \pi^2 > 0$ min
$f\left(\frac{3}{2}, \frac{3}{2}\right) = -\pi^2 < 0$ max
$f\left(\frac{3}{2}, \frac{5}{2}\right) = \pi^2 > 0$ min
$f\left(\frac{3}{2}, \frac{7}{2}\right) = -\pi^2 < 0$ max
etc.

Check some more value on your own.

On the grid below I'll use black dots for saddle points, red dots for maxs, and green dots for mins.