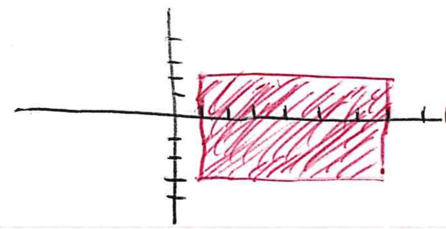
11.8 Lagrange Multipliers

Notation If A and B are sets, then $A \times B = \{(a,b) \mid a \in A, b \in B\}$.
Thus $[1,7] \times [-3,2] = \{(x,y) \mid 1 \le x \le 7, -3 \le y \le 2\}$
is the rectangle shown

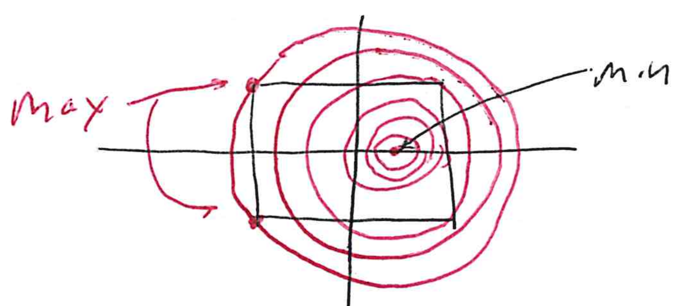


Ex Let $f(x,y) = x^2 - x + y^2$, $R = [-1,1] \times [-1,1]$.
Find the max and min of $f(x,y)$ over R.

Sol No calculus is required. Study the level curves.
Rewrite the function as

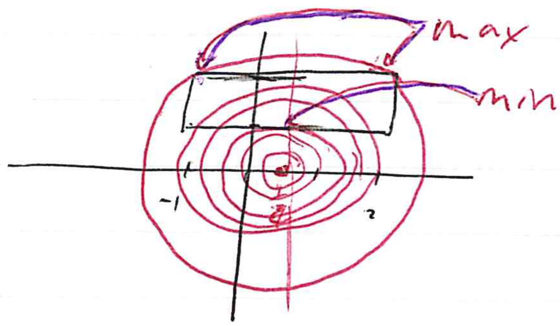
$$z = f(x,y) = (x - \tfrac{1}{2})^2 + y^2 - \tfrac{1}{4}.$$

The global min is a $(\frac{1}{2}, 0)$ and is $-\frac{1}{4}$.



Max's are at $(-1, \pm 1)$ and $f(-1,1) = f(-1,-1) = 3$.

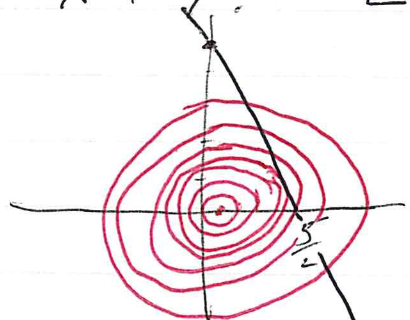
Ex Same $f(x,y) = x^2 - x + y^2$. $R = [-1,2] \times [1,2]$



Max is at the upper corners: $(-1,2)$ and $(2,2)$
where $f(2,2) = f(-1,2) = 6$.

Min is at the midpoint of bottom side where the level circle is tangent, $(\frac{1}{2}, 1)$
$f(\frac{1}{2}, 1) = 3/4$.

Ex Same $f(x,y) = x^2 - x + y^2$. $L := y = -2x + 5$.



No max. Min is at pt where level circle is tangent to L. How to find this pt?

Method I $\quad f(x,y) = f(x, -2x+5) = x^2 - x + (-2x+5)^2$

$$= x^2 - x + 4x^2 - 20x + 25$$

$$= 5x^2 - 21x + 25.$$

Compute $\frac{df}{dx}$ and set equal to 0.

$$\frac{df}{dx} = 10x - 21 = 0 \implies x = \frac{21}{10} = 2.1$$

Then $y = -2(2.1) + 5 = -4.2 + 5 = 0.8.$

Thus $\min = f(2.1, 0.8) = 2.95.$

Method II $\quad$ Let $g = y + 2x$. Then $g = 5$ is a level curve whose graph is L. We need tangent point. This happens when normal vectors are $\parallel$. That is when

$$\nabla f = \lambda \nabla g.$$

$$\langle 2x-1, 2y \rangle = \lambda \langle 2, 1 \rangle$$

$$2x - 1 = \lambda 2 \rightarrow x = \lambda + \tfrac{1}{2}$$

$$2y = \lambda \rightarrow y = \tfrac{1}{2}\lambda$$

$$2x + y = 5$$

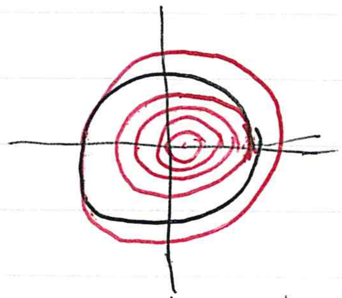
$$2(\lambda + \tfrac{1}{2}) + \tfrac{1}{2}\lambda = 5$$

$$\ldots \quad \lambda = 1.6$$

Thus $x = 2.1 \quad y = 0.8. \quad f(2.1, 0.8) = 2.95.$

The method does not say min or max. You have to use the picture

Ex Some $f(x,y) = x^2 - x + y^2$. Find extrema on $x^2+y^2=4$.



Method I Usually, we see the tangencies occur at $(-2,0)$ and $(2,0)$.
$f(-2,0) = 6 = \max \quad f(2,0) = 2$.

Method II Let $g(x,y) = x^2+y^2$. The constraint eq is $x^2+y^2=4$.
Then

$$\nabla g(x,y) = \langle 2x, 2y\rangle \qquad \nabla f = \langle 2x-1, 2y\rangle$$

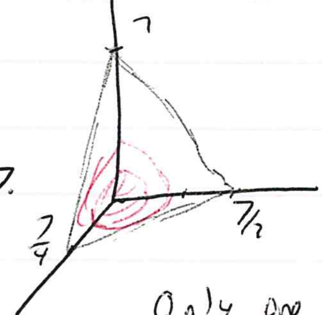
We want $\nabla f = \lambda \nabla g$

$$2x-1 = \lambda 2x$$
$$2y = \lambda 2y \Rightarrow \lambda = 1 \text{ or } y = 0.$$
$$x^2+y^2 = 4.$$

But if $\lambda = 1$
$-1 = 0$.
Thus $y=0$.
$x^2 = 4 \Rightarrow x = \pm 2$.

Ex Let $f(x,y,z) = 3x^2+2y^2+z^2$. Find the extrema and their locations of f subject to the constraint $4x+2y+z=7$. Are they mins or maxs.

Sol Let $g(x,y,z) = 4x+2y+z$.
So, the constraint eq. is $g(x,y,z)=7$.



Only one extrema. It will be a min.

$\nabla f = \langle 6x, 4y, 2z \rangle$
$\nabla g = \langle 4, 2, 1 \rangle$

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g = 7.$$

$$6x = 4\lambda \Rightarrow x = \tfrac{2}{3}\lambda$$
$$4y = 2\lambda \Rightarrow y = \tfrac{1}{2}\lambda$$
$$2z = \lambda \Rightarrow z = \tfrac{1}{2}\lambda$$
$$4x+2y+z=7$$

$$4\left(\tfrac{2}{3}\lambda\right)+2\left(\tfrac{1}{2}\lambda\right)+\tfrac{1}{2}\lambda = 7$$

$$\underbrace{\left(\tfrac{8}{3}+\overset{3}{\cancel{1}}+\tfrac{1}{2}\right)}\lambda = 7$$

$$\left(\frac{16+6+3}{6}\right)\lambda = 7$$

$$\lambda = 7\cdot\frac{6}{25} = \frac{42}{25}.$$

$$x = \frac{28}{25} \quad y = \frac{21}{25} \quad z = \frac{21}{25}$$

$$f\left(\tfrac{28}{25}, \tfrac{21}{25}, \tfrac{21}{25}\right) = \frac{3675}{625} = \frac{147}{25} = \boxed{5.88}$$

Ex Let $f(x,y) = x^2 + 2y^2 + 2xy + 2x + 3y$. Find the minim value of $f(x,y)$ when subject to the constraint $x^2 - y = 1$.

Sol We shall use the method of Lagrange multipliers. Let $g(x,y) = x^2 - y$. Then we need to solve:

$$\nabla f = \lambda \nabla g, \qquad (1)$$
$$g = 1, \qquad (2)$$

Eq (1) gives

$$2x + 2y + 2 = 2\lambda x, \qquad (3)$$
$$4y + 2x + 3 = -\lambda. \qquad (4)$$

Eliminating λ gives

$$x + y + 1 = -(4y + 2x + 3)x \qquad (5)$$

You can show that (5) and $y = 1 - x^2$ (from (2)) gives

$$x^2(4x + 3) = 0.$$

Thus, $x = 0$ or $x = -\frac{3}{4}$.

If $x = 0$, then $y = 0^2 - 1 = -1$. If $x = -\frac{3}{4}$, then $y = \left(-\frac{3}{4}\right) - 1 = -\frac{7}{16}$ (check this). Thus, there are two point we need to check: $(0,-1)$ and $\left(-\frac{3}{4}, -\frac{7}{16}\right)$.
You can check that

$$f(0,-1) = -1,$$
$$f\left(-\frac{3}{4}, -\frac{7}{16}\right) = -\frac{155}{128} = -1.2109375.$$

Hence, the min of $f(x,y)$ occurs at $\left(-3/4, -7/16\right)$.

Sol 2 This example can also be done by direct substitution

$$f(x, x^2-1) = x^2+2(x^2-1)^2+2x(x^2-1)+2x+3(x^2-1)$$

$$= 2x^4+2x^3-1$$

$$\frac{df}{dx} = 8x^3+6x^2 = 2x^2(4x+3).$$

This is zero for $x=0$ and $x=\frac{-3}{4}$.

You can use the one-variable second derivative test to show that $x=-3/4$ is a min and $x=0$ is an inflection point.

For $x=-3/4$, $y=-7/16$ and $f\left(\frac{-3}{4},\frac{-7}{16}\right)=\frac{-155}{128}$ as before

Ex The equation below determines a curve in the xy-plane.

$$x^2+2xy+2y^2=100.$$

Find the point ~~s~~ or points on this curve that are closest to the origin.

Warning The algebra will get tricky.

Sol. We will use the method of Lagrange multipliers. The function we need to minimize is $d(x,y)=\sqrt{x^2+y^2}$. But it is easier to work with the square, so we let

$$f(x,y)=x^2+y^2.$$

Then the constraint condition is

$$g(x,y)=x^2+2xy+2y^2=100. \qquad (1)$$

Now $\nabla f=\lambda\nabla g$ gives

$$2x=\lambda(2x+2y)$$
$$2y=\lambda(2x+4y)$$

This leads to $\frac{x}{x+y}=\lambda=\frac{y}{x+2y}$,

$$\frac{x+y}{x}=\frac{x+2y}{y}$$

$$1+\frac{y}{x}=\frac{x}{y}+2$$

$$\frac{y}{x}=\frac{x}{y}+1 \qquad (2)$$

Now we use a trick. Let $t = \frac{y}{x}$. Then (2) becomes

$$t = \frac{1}{t} + 1, \text{ or}$$

$$t^2 - t - 1 = 0.$$

Thus, $t = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$.

Case 1 Suppose $t = \frac{1+\sqrt{5}}{2}$. Rewrite equation (1) as

$$1 + 2t + 2t^2 = \frac{100}{x^2}.$$ (We divided thru by x^2.)

$$1 + 2\left(\frac{1+\sqrt{5}}{2}\right) + 2\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{100}{x^2}$$

$$2 + \sqrt{5} + \frac{1+2\sqrt{5}+5}{2} = \frac{100}{x^2}$$

$$5 + 2\sqrt{5} = \frac{100}{x^2}.$$

Thus, $x = \frac{\pm 10}{\sqrt{5+2\sqrt{5}}} \approx \pm 5.2573111$

Now $y = xt = \pm \frac{5(1+\sqrt{5})}{\sqrt{5+2\sqrt{5}}} \approx \pm 3.249197.$

For case 1 we have two points to check.

$$(5.2573111\ldots, 3.249197\ldots)$$

$$(-5.2573111\ldots, -3.249197\ldots)$$

Case 2 We leave the details to you.

$$x = \frac{\pm 10}{\sqrt{5-2\sqrt{5}}} \approx \pm 13.763815$$

$$y = tx = \frac{\pm 5 (1-\sqrt{5})}{\sqrt{5-2\sqrt{5}}} \approx \mp 8.5065081.$$

Thus there are two more points to check.

$$(13.763815\ldots, -8.5065081\ldots)$$

$$(-13.763815\ldots, 8.5065081\ldots)$$

Last step It is clear the first pair are closure to the origin than the second.

$$d\left(\frac{\pm 10}{\sqrt{5+2\sqrt{5}}}, \frac{\pm 5(1+\sqrt{5})}{\sqrt{5+2\sqrt{5}}}\right) \approx 6.180339888$$

$$d\left(\frac{\pm 10}{\sqrt{5-2\sqrt{5}}}, \frac{\mp 5(\sqrt{5}-1)}{\sqrt{5-2\sqrt{5}}}\right) \approx 16.180339888$$

Graph On the next page is a graph of $x^2+2xy+2y^2=100$. Of course you know that it is an ellipse. The two distance we found are half the lengths of the major and minor axes.

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> with(plots):with(plottools):
> gr:=implicitplot(x^2 + 2*x*y + 2*y^2 = 100,x=-15..15,y=-15..15,
  view=[-15..15,-15..15],numpoints=50000,color=black, thickness=2):
> cp:=contourplot(sqrt(x^2+y^2),x=-17..17,y=-17..17,view=[-17..17,
  -17..17],numpoints=50000,contours=[1,3,5,6.18,8,10,12,14,16.18],
  color=pink):
> display(gr,cp);
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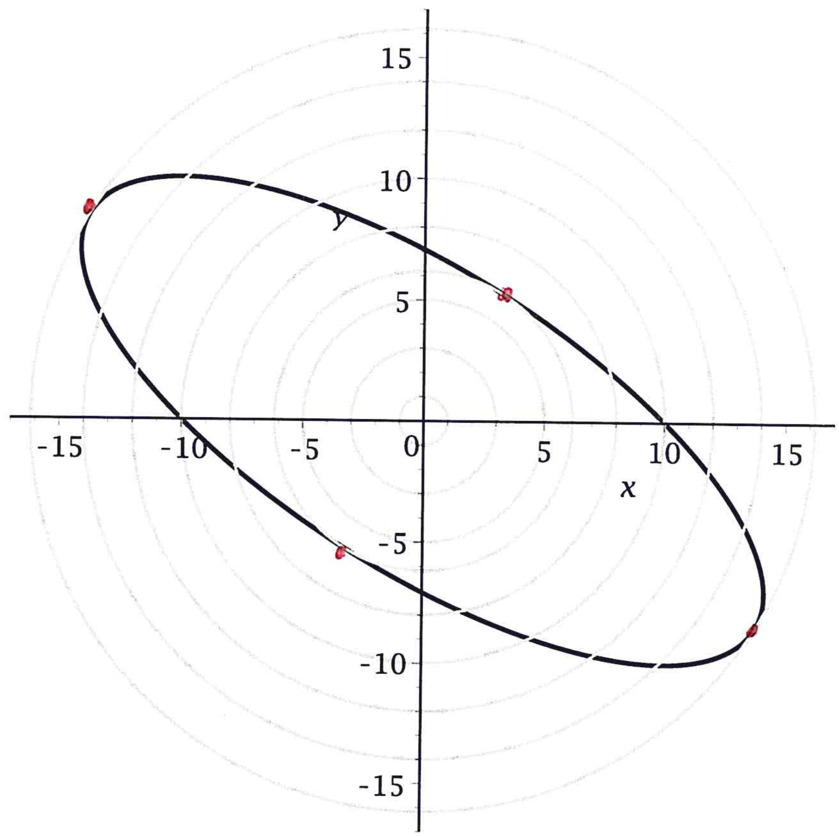


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