

Chapter 12 sections 12.1 & 12.2

Ex Find the volume under $f(x, y) = x + y^2$ over the rectangle $[1, 3] \times [2, 4]$.

Method I Use sums to approximate the volume

Let $1 = x_0 < x_1 < x_2 < \dots < x_m = 3$ and
 $2 = y_0 < y_1 < y_2 < \dots < y_n = 4$.

Assume they are equally spaced. Let $\Delta x = x_{i+1} - x_i = \frac{3-1}{m}$

Let $\Delta y = \frac{4-2}{n} = y_{i+1} - y_i$.

Then $\text{Vol} \approx \sum_{j=0}^{n-1} \left(\sum_{i=0}^{m-1} f(x_i, y_j) \Delta x \right) \Delta y$.

I used a computer (Maple) to do this for several choices of m and n . The number of calculations is very large. For $n = m = 1000$, there are 10^6 passes through the summation loop. It took a couple of minutes. Here are my results

$$m=n=5$$

$$39.84$$

$$m=40 \quad n=50$$

$$44.7544$$

$$m=n=100$$

$$45.0536$$

$$m=n=1000$$

$$45.305336$$

~~4/16~~

To get the exact answer we take the limit as $n, m \rightarrow \infty, \Delta x, \Delta y \rightarrow 0$. Then we get

$$\int_2^4 \int_1^3 f(x, y) dx dy$$

In our example

$$\int_2^4 \left(\int_1^3 x + y^2 dx \right) dy = \int_2^4 \left[\frac{x^2}{2} + xy^2 \right]_1^3 dy$$
$$= \int_2^4 \left[\frac{9}{2} + 3y^2 \right] - \left[\frac{1}{2} + y^2 \right] dy = \int_2^4 4 + 2y^2 dy$$

$$4y + \frac{2}{3}y^3 \Big|_2^4 = \left(16 + \frac{128}{3} \right) - \left(8 + \frac{16}{3} \right)$$

$$= 8 + \frac{112}{3} = \frac{24}{3} + \frac{112}{3} = \frac{136}{3} = 45 \frac{1}{3}$$

Find the approximate volume of $f(x,y)$ over the rectangle $[a,b] \times [c,d]$.

```
> f := (x,y) -> x + y^2;
```

$$f := (x, y) \rightarrow x + y^2$$

```
> a:=1: b:= 3: c:=2: d:=4: m:=5: n:=5: dx := (b-a)/m; dy := (d-c)/n;
```

$$dx := \frac{2}{5}$$

$$dy := \frac{2}{5}$$

```
> Volume:=0:
```

```
for i from 0 to m-1 do
```

```
  for j from 0 to n-1 do Volume := Volume + f(a+i*dx,c+j*dy)*dx*dy
```

```
  end do
```

```
end do;
```

```
evalf(Volume);
```

39.84000000

```
> m:=40:n:=50:dx := (b-a)/m; dy := (d-c)/n;
```

$$dx := \frac{1}{20}$$

$$dy := \frac{1}{25}$$

```
> Volume:=0:
```

```
for i from 0 to m-1 do
```

```
  for j from 0 to n-1 do Volume := Volume + f(a+i*dx,c+j*dy)*dx*dy
```

```
  end do
```

```
end do;
```

```
evalf(Volume);
```

44.75440000

```
> m:=100:n:=100:dx := (b-a)/m; dy := (d-c)/n;
```

$$dx := \frac{1}{50}$$

$$dy := \frac{1}{50}$$

```
> Volume:=0:
```

```
for i from 0 to m-1 do
```

```
  for j from 0 to n-1 do Volume := Volume + f(a+i*dx,c+j*dy)*dx*dy
```

```
end do
end do;
evalf(Volume);
```

45.05360000

```
> m:=1000:n:=1000:dx := (b-a)/m; dy := (d-c)/n;
```

$$dx := \frac{1}{500}$$

$$dy := \frac{1}{500}$$

```
> Volume:=0:
for i from 0 to m-1 do
  for j from 0 to n-1 do Volume := Volume + f(a+i*dx,c+j*dy)*dx*dy
  end do
end do;
evalf(Volume);
```

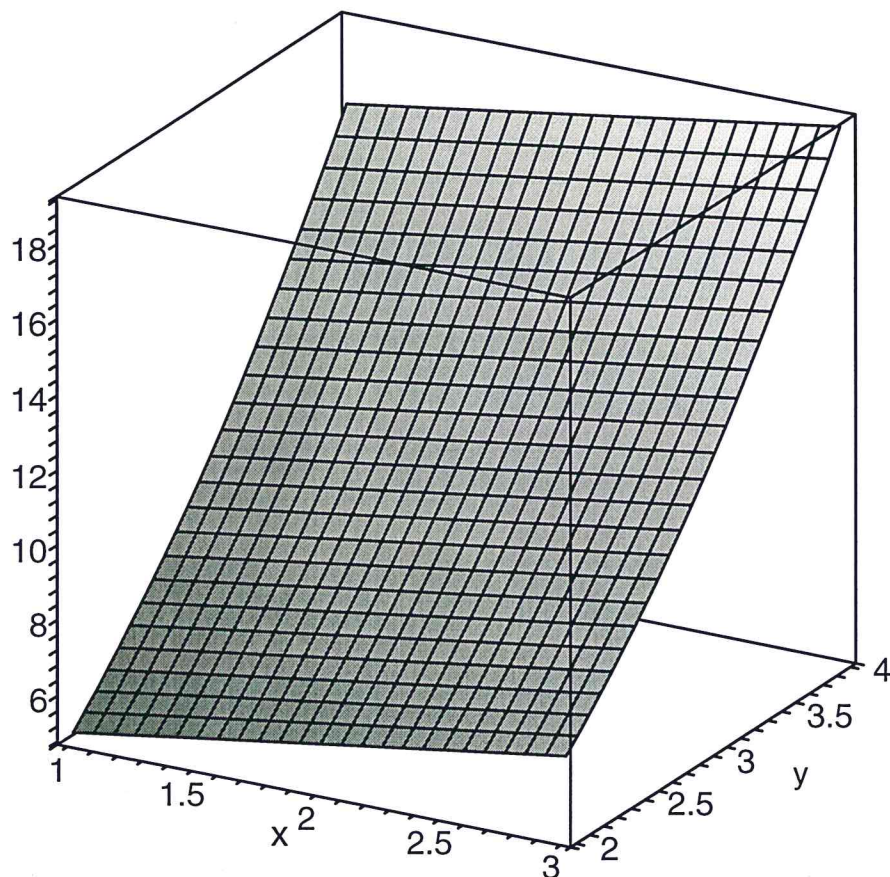
45.30533600

Now we compare to the double integral.

```
> evalf(int(int(x+y^2,x=1..3),y=2..4));
```

45.33333333

```
> plot3d(x+y^2,x= 1..3,y= 2..4);
```



Ex Find the volume under $z = x^2y + xy^2$
over $[1, 2] \times [3, 6]$

$$V = \int_3^6 \int_1^2 x^2y + xy^2 \, dx \, dy$$

$$= \int_3^6 \left. \left(\frac{x^3y}{3} + \frac{x^2y^2}{2} \right) \right|_1^2 \, dy$$

$$= \int_3^6 \left[\frac{8}{3}y + 2y^2 \right] - \left[\frac{1}{3}y + \frac{1}{2}y^2 \right] \, dy$$

$$= \int_3^6 \left(\frac{7}{3}y + \frac{3}{2}y^2 \right) \, dy = \left. \frac{7}{6}y^2 + \frac{1}{2}y^3 \right|_3^6$$

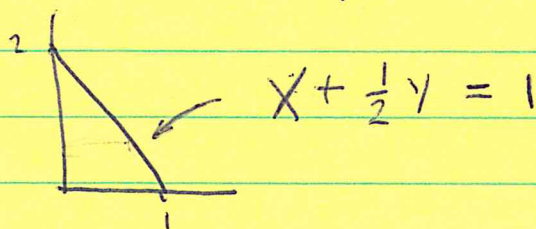
$$= (42 + 108) - \underbrace{\left(\frac{21}{2} + \frac{27}{2} \right)}_{16}$$

$$= 150 - 16 = 134$$

$$\boxed{\text{Volume} = 134}$$

Ex

Find Volume under $z = x^2 y^2$ over the triangle



Let's do $dx dy$. Then $V = \int_0^2 \int_0^{x = -\frac{1}{2}y + 1} x^2 y^2 dx dy$

$$= \int_0^2 \left. \frac{1}{3} x^3 y^2 \right|_0^{-\frac{1}{2}y + 1} dy = \int_0^2 \frac{1}{3} \left(1 - \frac{1}{2}y\right)^3 y^2 dy$$

$$= \frac{1}{3} \int_0^2 \left(1 - \frac{3}{2}y + \frac{3}{4}y^2 - \frac{1}{8}y^3\right) y^2 dy$$

$$= \frac{1}{3} \int_0^2 y^2 - \frac{3}{2}y^3 + \frac{3}{4}y^4 - \frac{1}{8}y^5 dy =$$

$$= \frac{1}{3} \left[\frac{1}{3}y^3 - \frac{3}{8}y^4 + \frac{3}{20}y^5 - \frac{1}{48}y^6 \right]_0^2$$

$$= \frac{1}{3} \left[\frac{8}{3} - 6 + \frac{24}{5} - \frac{4}{3} \right] = \frac{1}{45} \left[\begin{array}{l} 40 - 90 + 72 - 20 \\ 112 - 110 \end{array} \right]$$

$$= \frac{2}{45} = .0\bar{4}$$

Ex Same problem, but now do $dy dx$

$$\int_0^1 \int_0^{2-2x} x^2 y^2 dy dx$$

$$= \int_0^1 \left. \frac{1}{3} x^2 y^3 \right|_0^{2-2x} dx = \int_0^1 \frac{1}{3} x^2 (2-2x)^3 dx$$

$$= \frac{8}{3} \int_0^1 x^2 (1-3x+3x^2-x^3) dx$$

$$= \frac{8}{3} \int_0^1 x^2 - 3x^3 + 3x^4 - x^5 dx$$

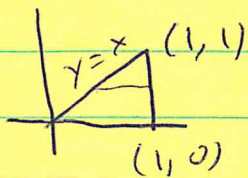
$$= \frac{8}{3} \left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6} \right)$$

$$= \frac{8}{3} \left(\frac{-5}{12} + \frac{13}{30} \right)$$

$$= \frac{8}{3} \left(\frac{-25+26}{60} \right) = \frac{8}{3} \frac{1}{60} = \frac{2}{3 \cdot 15} = \frac{2}{45} = .0\bar{4}$$

Ex

Find the volume under $z = ye^{x^3}$ over the triangle



First we try $dx dy$

$$V = \int_0^1 \int_y^1 ye^{x^3} dx dy = ? \text{ stuck?}$$

Next try $dy dx$

$$V = \int_0^1 \int_0^x ye^{x^3} dy dx$$

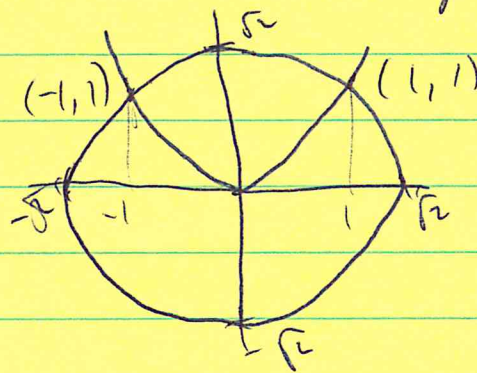
$$= \int_0^1 \left. \frac{1}{2} y^2 e^{x^3} \right|_0^x dx$$

$$= \int_0^1 \frac{1}{2} x^2 e^{x^3} dx \quad u = x^3 \quad du = 3x^2 dx$$

$$= \frac{1}{6} \int_0^1 e^u du = \frac{1}{6} e^u \Big|_0^1 = \frac{1}{6} (e - 1)$$

$$\approx 0.286380305$$

Ex Find the area of the region $R = \{(x, y) \mid x^2 + y^2 \leq 2 \text{ and } y \geq x^2\}$



1. Find intersection points. Sub. $y = x^2$ into $x^2 + y^2 = 2$ to get $x^2 + x^2 - 2 = 0$. So $y = 1$ or -2 . Only $y = 1$ gives real solutions for x . These points are $(-1, 1)$ and $(1, 1)$.

2. Think of $f(x, y) = 1$. Do $dy dx$

$$A = \int_{-1}^1 \int_{x^2}^{\sqrt{2-x^2}} 1 \, dy \, dx$$

$$= \int_{-1}^1 \sqrt{2-x^2} - x^2 \, dx = 2 \int_0^1 \sqrt{2-x^2} \, dx - 2 \int_0^1 x^2 \, dx$$

$= -2/3$

$$\int_0^1 \sqrt{2-x^2} \, dx = \int_0^{\pi/4} 2 \cos^2 \theta \, d\theta$$

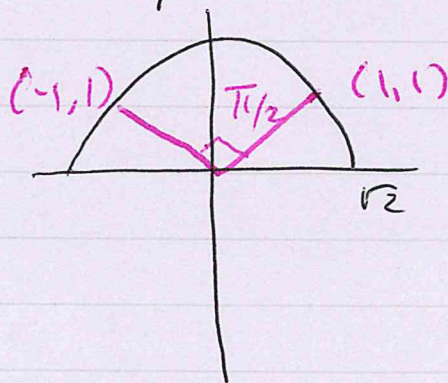
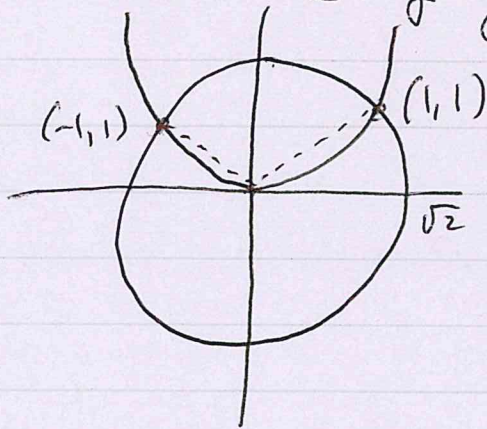
Let $x = \sqrt{2} \sin \theta$

Then $dx = \sqrt{2} \cos \theta \, d\theta = \int_0^{\pi/4} \frac{1 + \cos(2\theta)}{2} \, d\theta$

$$= \left(\theta + \frac{1}{2} \sin(2\theta) \right) \Big|_0^{\pi/4} = \frac{\pi}{4} + \frac{1}{2} \cdot 1 - (0 + 0) = \frac{\pi}{4} + \frac{1}{2}$$

$$A = 2 \left(\frac{\pi}{4} + \frac{1}{2} \right) - \frac{2}{3} = \frac{\pi}{2} + \frac{1}{3}$$

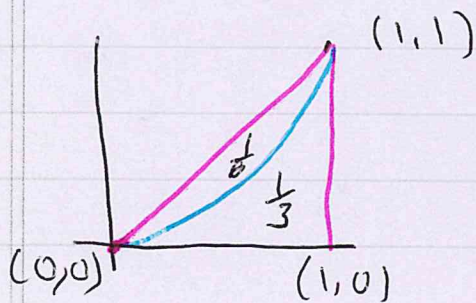
Solution using geometry.



Area of circle is $\pi r^2 = 2\pi$.

Area of one quarter sector is $\frac{1}{4}\pi r^2 = \frac{\pi}{2}$.

We just need to add on the two "slivers" between the dotted lines and $y=x^2$.



Area of triangle is $\frac{1}{2}$.

Area under $y=x^2$ is $\int_0^1 x^2 dx = \frac{1}{3}$.

Thus, area of "sliver" is $\frac{1}{6}$.

Total area of region is $\frac{\pi}{2} + \frac{1}{6} + \frac{1}{6} = \frac{\pi}{2} + \frac{1}{3}$

Ex

Find the volume under $f(x,y) = 7+x+2y$ over R from last time.

$$\int_{-1}^1 \int_{x^2}^{\sqrt{2-x^2}} 7+x+2y \, dy \, dx$$

$$= \int_{-1}^1 (7y + xy + y^2) \Big|_{x^2}^{\sqrt{2-x^2}} \, dx$$

$$= \int_{-1}^1 [7\sqrt{2-x^2} + x\sqrt{2-x^2} + (2-x^2)] - [7x^2 + x^3 + x^4] \, dx$$

$$\int_{-1}^1 7\sqrt{2-x^2} + x\sqrt{2-x^2} - x^4 - x^3 - 8x^2 + 2 \, dx$$

$$= 7\left(\frac{\pi}{2} + 1\right) - \frac{2}{5} - 8 \cdot \frac{2}{3} + 4$$

$$= \frac{7\pi}{2} + 11 - \frac{2}{5} - \frac{16}{3}$$

$$= \frac{7\pi}{2} + \frac{185}{15} - \frac{26}{15} - \frac{80}{15} = \boxed{\frac{7\pi}{2} + \frac{79}{15}} \quad \begin{matrix} 15 \\ 165 \\ -86 \\ \hline 79 \end{matrix}$$

`int(int(7+x+2*y,y=x^2..sqrt(2-x^2)),x=-1..1);`

$$\frac{7}{2}\pi + \frac{79}{15}$$

